Math 106: Calculus

Final - Fall 2009

Duration : 180 minutes

#1	10	
#2	10	
#3	15	
#4	10	
#5	10	
#6	10	
#7	15	
#8	10	
#9	10	
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Name ____________________

Student ID ____________________

Signature ____________________

- Put your name, student ID and signature in the boxes above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  Section 1 (Sultan Erdoğan Demir, MW 11:30-13:20) ____________________

  Section 2 (Sultan Erdoğan Demir, MW 14:30-16:20) ____________________

  Section 3 (Emre Mengi, MW 9:30-11:20) ____________________

  Section 4 (Emre Mengi, MW 14:30-16:20) ____________________

  Section 5 (Kazim Büyükboduk, TuTh 11:30-13:20) ____________________

  Section 6 (Kazim Büyükboduk, TuTh 14:30-16:20) ____________________

Question 1. Determine whether each of the following series is convergent or divergent. Explain your answer fully.

(a) $\sum_{n=2}^{\infty}(-1)^n \frac{\sqrt[3]{n}}{\ln n}$

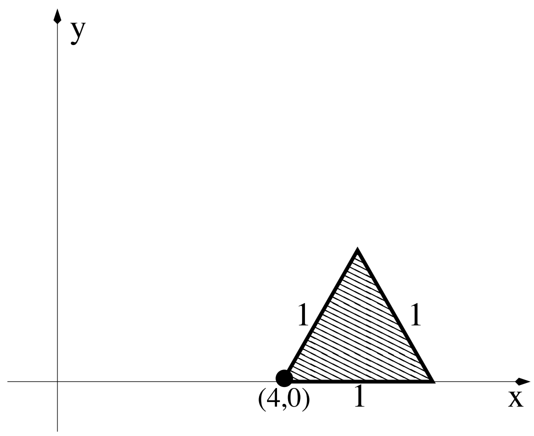
(b) $\sum_{n=1}^{\infty} \frac{\cos\sqrt{n}}{n^3}$

(c) $\sum_{n=1}^{\infty} \sin\left(\frac{\pi}{n^3}\right)$

Question 2. In (a) and (b) below, find the indicated area or volume by first expressing it as a definite integral, and then evaluating the definite integral.

(a) The area of the region between $x = y^2 - 6y$ and $x = 4y - y^2$.

(b) The <u>volume</u> obtained by rotating the equilateral triangle shown in the figure below about the y-axis.
(Remark: The equilateral triangle lies above the x-axis except its base which lies on the x-axis. Each side of the equilateral triangle is of length 1. The left-most corner of the equilateral triangle has coordinates $(4, 0)$.)



Question 3.

(a) Evaluate the limit $\displaystyle \lim_{x\to 0} \frac{x \cdot \int_0^x \tan(t^2)\, dt}{\sin(x^2)}$.

(b) Find the function defined by

$$F(t) = \int_{\sqrt{t}}^{t} \frac{d}{dx}\left(e^{x^{2x}}\right)\, dx$$

for all $t \geq 0$. Your answer should <u>not</u> involve an integral nor a derivative.

(c) Find the function defined by

$$G(t) = \frac{d}{dt}\left(\int_{\sqrt{t}}^{t} e^{x^{2x}}\, dx\right)$$

for all $t > 0$. Your answer should <u>not</u> involve an integral or a derivative.

Question 4. Prove that the polynomial $P(x) = x^3 + 2x + 3$ has exactly one root in $(-\infty, \infty)$.

Question 5.

(a) Estimate the integral

$$\int_0^4 3^{\sqrt{x}}\,dx$$

using a right-sum (i.e., the heights of the rectangles are given by the values of the function at the right end-points) with $n = 4$ rectangles of width $\Delta x = 1$. Is your estimate an upper bound or a lower bound for the exact integral? Explain.

(b) Evaluate the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\left(1+\frac{i}{n}\right)\ln\left(1+\frac{i}{n}\right)}{n}$$

by interpreting it as a definite integral and then calculating the value of the integral.

Question 6. Compute the following integrals. Show all your reasoning clearly.

(a) $\displaystyle\int_0^{\pi/2} \sin^4(x)\cos^3(x)\, dx$

(b) $\displaystyle\int \frac{1}{x^2\sqrt{36-x^2}}\, dx$

Question 7.

(a) Find the Taylor series $T(x)$ for $\cos x$ centered at $\pi/3$.

(b) Show that the Taylor series $T(x)$ that you determined in part **(a)** satisfies $\cos x = T(x)$ for all $x \in (-\infty, \infty)$.

Question 8.

(a) Find the radius and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n(n+1)}$$

(b) Newton discovered that

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2}\, x^n$$

for $-1 < x < 1$.

(i) Using this formula, find a power series expansion for $\arcsin x$.

(ii) Use your power series from part **(i)** with $x = 1/\sqrt{2}$ to find a power series whose sum is π.

Question 9. Determine whether the following improper integrals are convergent or divergent. Evaluate them when they are convergent. Show all your reasoning.

(a) $\displaystyle\int_1^\infty \frac{1}{(x+2)(x+3)(x+4)}\,dx$

(b) $\displaystyle\int_{-1}^1 \frac{1}{x^{4/3}}\,dx$