LECTURE 14

DUALITY FOR SEMIDEFINITE PROGRAMS

Standard inner product in $\mathbb{R}^n$

$$\langle x, y \rangle = x^T y$$

Euclidean norm

$$\|x\|_2 = \sqrt{\langle x, x \rangle} = \sqrt{x^T x}$$

Standard inner product in $\mathbb{R}^{n \times n}$

$$\langle X, Y \rangle = \text{Trace}(X^T Y)$$

Euclidean norm (Frobenius norm)

$$\|X\|_F = \sqrt{\langle X, X \rangle} = \sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} x_{jk}^2}$$

S^n — set of $n \times n$ symmetric matrices (subspace of $\mathbb{R}^{n \times n}$)

S^n_+ — set of $n \times n$ symmetric PSD matrices (closed convex cone)

S^n_{++} — set of $n \times n$ symmetric PD matrices

Semidefinite program

$$(\text{PSDP}) \quad \begin{array}{ll} \text{minimize} & \langle C, X \rangle \\ & \langle A_j, X \rangle = b_j \quad j = 1, \ldots, m \\ & X \in S^n_+ \end{array}$$

$A_1, \ldots, A_m, C \in S^n_+$ - given

$b \in \mathbb{R}^m$

This is a convex program:

* objective $\langle C, X \rangle$ is linear, thus convex;
* the feasible region

$$F = \{ X \mid \langle A_j, X \rangle = b_j \quad j = 1, \ldots, m, \underbrace{\lambda_k(X) \geq 0 \quad k = 1, \ldots, n}_{\text{OR } X \text{ is PSD}} \}$$

is convex. (Here $\lambda_k(X)$ denotes the kth largest eigenvalue of X.)

Dual problem

$$L(x; \mu, \phi) := \langle C, X \rangle - \sum_{j=1}^{m} \mu_j (\langle A_j, X \rangle - b_j) - \phi \, \lambda_{min}(X)$$

$$\underline{g}(\mu, \phi) := \inf_{X \in S_+^n} L(X; \mu, \phi)$$

$$= \inf_{X \in S_+^n} \langle C, X \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X \rangle - \phi \cdot \lambda_{min}(X) + \mu^T b$$

Notice for each $\mu \in \mathbb{R}^m$, nonnegative $\phi \in \mathbb{R}$

$$\underline{g}(\mu, \phi) = \begin{cases} b^T \mu & \text{if } \langle C, X \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X \rangle \geq \phi \cdot \lambda_{min}(X) \quad \forall X \in S_+^n \\ -\infty & \text{otherwise} \end{cases}$$

Consequently,

$$\begin{array}{ll} \underset{\substack{\mu \in \mathbb{R}^m, \phi \in \mathbb{R} \\ \phi \geq 0}}{\text{maximize}} & \underline{g}(\mu, \phi) \end{array}$$

$$= \begin{array}{ll} \underset{\mu \in \mathbb{R}^m}{\text{maximize}} & b^T \mu \\ & \exists \phi \geq 0 \; \forall X \in S_+^n \; \langle C, X \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X \rangle \geq \phi \cdot \lambda_{min}(X) \end{array}$$

CLAIM

$$\left\{ C - \sum_{j=1}^{m} \mu_j A_j \;\middle|\; \exists \phi \; \forall X \in S_+^n \; \langle C, X \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X \rangle \geq \phi \cdot \lambda_{min}(X) \right\}$$

$$=$$

$$\left\{ C - \sum_{j=1}^{m} \mu_j A_j \;\middle|\; \right.$$

CLAIM

The following are equivalent for a given $\mu \in \mathbb{R}^m$:

(i) $\exists \phi \geq 0 \quad \forall X \in S^n_+$

$$\langle C, X \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X \rangle \geq \phi \cdot \lambda_{min}(X)$$

(ii) $\left(C - \sum_{j=1}^{m} \mu_j A_j \right) \in S^n_+$

Proof

(i) $\Longrightarrow$ (ii)

Let $q_1, \ldots, q_n$ be eigenvectors of $C - \sum_{j=1}^{m} \mu_j A_j$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$.
Notice that $q_j q_j^T \in S^n_+$ for $j = 1, \ldots, n$.
Thus

$$\langle C, q_j q_j^T \rangle - \sum_{j=1}^{m} \mu_j \langle A_j, q_j q_j^T \rangle \geq 0$$

$$\Longrightarrow \quad q_j^T \left(C - \sum_{j=1}^{m} \mu_j A_j \right) q_j \geq 0$$

$$\Longrightarrow \quad \lambda_j \|q_j\|^2 \geq 0 \quad \Longrightarrow \quad \lambda_j \geq 0.$$

Consequently, $\left(C - \sum_{j=1}^{m} \mu_j A_j \right) \in S^n_+$. $\square$

(ii) $\Longrightarrow$ (i)

Let $\phi := \lambda_{min}\left(C - \sum_{j=1}^{m} \mu_j A_j\right) \geq 0$.

For all $X \in S_+^n$, we have

$$\langle C, X\rangle - \sum_{j=1}^{m} \mu_j \langle A_j, X\rangle \geq$$

$$\lambda_{min}\left\{X^{1/2}\left(C - \sum_{j=1}^{m} \mu_j A_j\right) X^{1/2}\right\}.$$

Denoting the unit eigenvectors of X by ~~$\nearrow$~~ $v_1, \ldots, v_n$ corresponding to the eigenvalues $n_1, \ldots, n_n$, in nonincreasing order

we have ~~for each un~~

$$\lambda_{min}\left\{X^{1/2}\left(C - \sum_{j=1}^{m} \mu_j A_j\right) X^{1/2}\right\}$$

$$= \min_{\substack{z \in \mathbb{R}^n \\ \|z\| = 1}} z^T X^{1/2}\left(C - \sum_{j=1}^{m} \mu_j A_j\right) X^{1/2} z$$

$$= \min_{\substack{\alpha_1, \ldots, \alpha_n \in \mathbb{R} \\ \|\alpha\| = 1}} (\alpha_1 v_1 + \cdots + \alpha_n v_n)^T X^{1/2}\left(C - \sum_{j=1}^{m} \mu_j A_j\right) X^{1/2} (\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$\geq \min_{\substack{\alpha_1, \ldots, \alpha_n \in \mathbb{R} \\ \|\alpha\| = 1}} n_1 \alpha_1^2 \phi + \cdots + n_n \alpha_n^2 \phi$$

$$\geq n_n \phi = \lambda_{min}(X) \cdot \phi.$$

We obtain

$$\text{(DSDP)} \quad \begin{aligned} &\text{maximize} \quad b^T \mu \\ &\mu \in \mathbb{R}^m \\ &\left(C - \sum_{j=1}^{m} \mu_j A_j\right) \in S^n_+ \end{aligned}$$

<u>THM</u>

Let p_*, d_* be optimal values for (PSDP), (DSDP), respectively. Following holds:

(i) $p_* \geq d_*$;

slater condition

(ii) If $\exists X \in S^n_{++}$ s.t. $\langle A_j, X \rangle = b_j$ $j = 1, \ldots, m$, then we have $p_* = d_*$.