LECTURE 9

LAGRANGIAN DUALITY

(Primal problem)
$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \\ &c_j(x) = 0 \quad j \in E \\ &c_j(x) \geq 0 \quad j \in I \end{aligned}$$

$$f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$$

$$c_j : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$$

$$\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$$
$$\text{dom } c_j := \{x \in \mathbb{R}^n \mid c_j(x) > -\infty\}$$

$$D := \left\{\bigcap_{j \in E \cup I} \text{dom } c_j\right\} \cap \text{dom } f$$

$$F := \{x \in D \mid c_j(x) = 0 \ \forall j \in E, \ c_j(x) \geq 0 \ \forall j \in I\}$$

Optimal value of the primal problem $p_* \in \mathbb{R} \cup \{-\infty, \infty\}$.

$p_* := \infty$ if $F = \emptyset$

$p_* := -\infty$ if $\exists \{x^{(k)}\}$ such that

(i) $x^{(k)} \in F \quad \forall k$

(ii) $\lim_{k \to \infty} f(x^{(k)}) = -\infty$.

Otherwise

$$p_* := \inf \{ f(x) \mid x \in F \}$$

Lagrange dual function

$$g(\mu, \lambda) := \inf_{x \in D} L(x; \mu, \lambda)$$

where

$$L(x; \mu, \lambda) := f(x) - \sum_{j \in E} \mu_j c_j(x) - \sum_{j \in I} \lambda_j c_j(x).$$

Ex

①
$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad x^T W x \\ &Ax = b \end{aligned}$$

W - symmetric PD $n \times n$ matrix

A - $m \times n$ matrix with $m < n$

$b \in \mathbb{R}^n$

$$L(x;\mu) = x^T W x - \mu^T (Ax - b)$$

$$g(\mu) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \; x^T W x - \mu^T (Ax - b)$$

The minimizer x_* satisfies

$$2W x_* - A^T \mu = 0$$

$$x_* = \frac{1}{2} W^{-1} (A^T \mu)$$

Thus

$$g(\mu) = \frac{1}{4} \mu^T A W^{-1} A^T \mu - \frac{1}{2} \mu^T A W^{-1} A^T \mu + b^T \mu$$

$$= -\frac{1}{4} \mu^T A W^{-1} A^T \mu + b^T \mu$$

~~2 minimize $x^T W x$~~

~~$x \in \mathbb{R}^n$~~

~~$x_j^2 = 1 \quad j = 1, \ldots, n$~~

2 $\quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad c^T x$

$$Ax = b$$

$$x \geq 0$$

$$L(x; \mu, \lambda) = c^T x - (Ax - b)^T \mu - x^T \lambda$$

For $\lambda \geq 0$

$$g(\mu, \lambda) = \underset{x \in \mathbb{R}^n}{\text{minimize}} \left(c - A^T \mu - \lambda\right)^T x + b^T \mu$$

Consequently,

$$g(\mu,\lambda) = \begin{cases} b^T\mu & \text{if } c = A^T\mu + \lambda \\ -\infty & \text{otherwise} \end{cases}$$

For each $\tilde{x} \in F$ and each $\lambda \geq 0$, μ

$$L(\tilde{x};\mu,\lambda) = f(\tilde{x}) - \sum_{j\in E} \mu_j \underbrace{c_j(\tilde{x})}_{0} - \underbrace{\sum_{j\in I} \lambda_j c_j(\tilde{x})}_{\geq 0}$$

$$\leq f(\tilde{x})$$

$$\Longrightarrow \underbrace{g(\mu,\lambda)}_{= \inf L(\tilde{x};\mu,\lambda)} \leq f(\tilde{x})$$

Lagrange dual problem

$$\begin{aligned} &\underset{\mu\in\mathbb{R}^m,\ \lambda\in\mathbb{R}^p}{\text{maximize}} \quad g(\mu,\lambda) && \left(\begin{array}{l} |E| = m \\ |I| = p \end{array}\right) \\ &\quad \lambda \geq 0 && \left(\begin{array}{l} g : \mathbb{R}^m\times\mathbb{R}^p \to \mathbb{R} \\ \text{dom } g := \{(\mu,\lambda) \mid g(\mu,\lambda) > -\infty\} \end{array}\right) \end{aligned}$$

$d_* := \infty$ if $\exists \{(\mu^{(k)},\lambda^{(k)})\}$ such that

(i) $\lambda^{(k)} \geq 0 \quad \forall k$

(ii) $\lim_{k\to\infty} g(\mu^{(k)},\lambda^{(k)}) = \infty$.

$d_* := -\infty$ if $\text{dom } g \cap \{(\mu,\lambda) \mid \lambda \geq 0\} = \emptyset$

otherwise

$$d_* := \sup \{ g(\mu, \lambda) \mid (\mu, \lambda) \in \operatorname{dom} g \text{ and } \lambda \geq 0 \}$$

Weak duality

$$d_* \leq p_*$$

always holds.

Ex

(1)
$$\underset{\substack{x \in \mathbb{R}^n \\ Ax = b}}{\text{minimize}} \quad x^T W x$$

dual problem

$$\underset{\mu \in \mathbb{R}^m}{\text{maximize}} \quad b^T \mu - \frac{1}{4} \mu^T A W^{-1} A^T \mu$$

(2)
$$\underset{\substack{x \in \mathbb{R}^n \\ Ax = b \\ x \geq 0}}{\text{minimize}} \quad c^T x$$

Dual problem

$$\underset{\substack{\mu \in \mathbb{R}^m,\ \lambda \in \mathbb{R}^n \\ \lambda \geq 0}}{\text{maximize}} \begin{cases} b^T\mu & \text{if } c = A^T\mu + \lambda \\ -\infty & \text{otherwise} \end{cases}$$

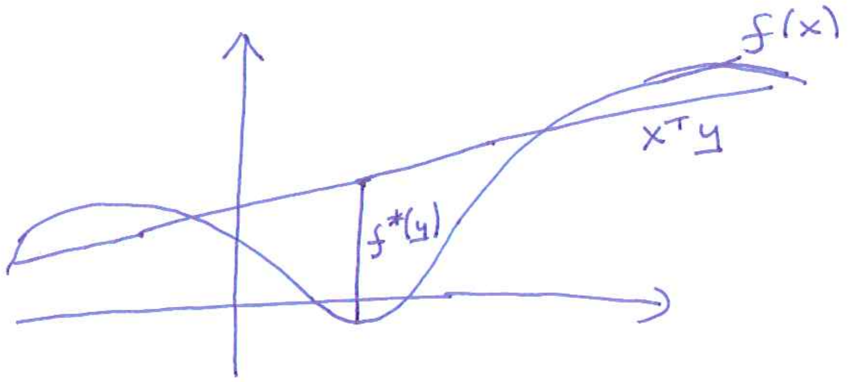
equivalently

$$\underset{\substack{\mu \in \mathbb{R}^m,\ \lambda \in \mathbb{R}^n \\ A^T\mu + \lambda = c \\ \lambda \geq 0}}{\text{maximize}} \quad b^T\mu$$

Linear constraints and Fenchel conjugate

Fenchel conjugate of f

$$f^*(\underline{y}) := \sup_{x \in \operatorname{dom} f} \{ x^T \underline{y} - f(x) \}$$



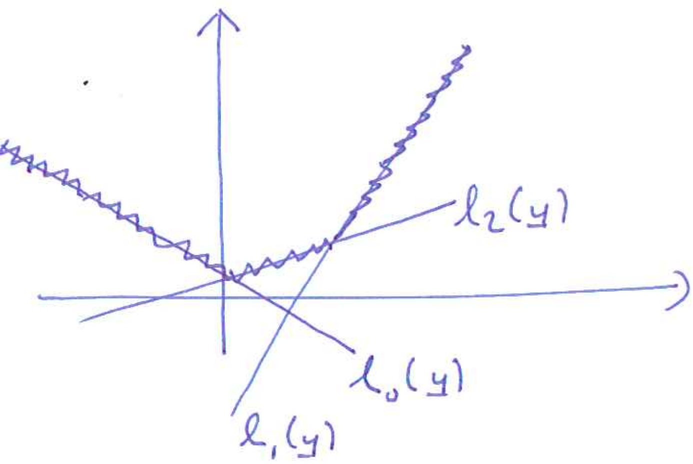
convex, that is

$$f^*(\theta \underline{y}_1 + (1-\theta)\underline{y}_2) =$$

$$\sup_{x \in \operatorname{dom} f} \left[x^T \theta \underline{y}_1 - \theta f(x) \right] + \left[x^T (1-\theta) \underline{y}_2 - (1-\theta) f(x) \right]$$

$$\leq \theta \sup_{x \in \operatorname{dom} f} \{ x^T \underline{y}_1 - f(x) \} + (1-\theta) \sup_{x \in \operatorname{dom} f} \{ x^T \underline{y}_2 - f(x) \}$$

$$= \theta \, f^*(\underline{y}_1) + (1-\theta) f^*(\underline{y}_2)$$



$$l_x(\underline{y}) = x^T \underline{y} - f(x)$$

~~Ex~~

$$\cancel{f(x) = x \cdot \ln x}$$

$$\cancel{f^*(y) = \sup_{x > 0} x^T \underline{y} - x \ln x}$$

$$\cancel{y > 0} \qquad \cancel{f^*(y) \neq \infty}$$

Ex

(1) $f(x) = \frac{1}{2} x^T W x$

$$f^*(\underline{y}) = \sup_x \underbrace{x^T \underline{y} - \frac{1}{2} x^T W x}_{(*)}$$

If W is not ~~PSD PD~~ PSD

$$f^*(\underline{y}) = \infty$$

(letting $x(t) = t v$ where v corresponds to a negative eigval of W

$$x(t)^T \underline{y} - \frac{1}{2} x(t)^T W x(t) \longrightarrow \infty$$

as $t \to \infty$)

If W is ~~PSD~~ PD

$$x_* = W^{-1} \underline{y}$$

is the maximizer of $(*)$. Thus

$$f^*(\underline{y}) = \frac{1}{2} \underline{y}^T W^{-1} \underline{y}$$

(W is PSD and singular - exercise)

$$f(x) = \|x\|$$

$$f^*(\underline{y}) = \sup_{x} \underline{y}^T x - \|x\|$$

Dual norm of $\|\cdot\|$

$$\|\underline{y}\|_* := \sup \{ \underline{y}^T x \mid \|x\| \leq 1 \}$$

Norm	Dual norm
$\|\cdot\|_2$	$\|\cdot\|_2$
$\|\cdot\|_\infty$	$\|\cdot\|_1$
$\|\cdot\|_1$	$\|\cdot\|_\infty$
$\|\cdot\|_p$	$\|\cdot\|_q$ where q is s.t. $\frac{1}{p} + \frac{1}{q} = 1$

Suppose $\|\underline{y}\|_* > 1$, there exists z such that

$$\|z\| \leq 1 \quad \text{and} \quad z^T \underline{y} > 1.$$

Letting $x(t) = t z$

$$\underline{y}^T x(t) - \|x(t)\| =$$

$$t (z^T \underline{y}) - t \|z\| =$$

$$t (z^T \underline{y} - \|z\|) \to \infty \quad \text{as} \quad t \to \infty.$$

Consequently,

$$f^*(\underline{y}) = \infty.$$

On the other hand, if $\|y\|_* \leq 1$,

for each x

$$x^T y \leq \|x\| \, \|y\|_* .$$

Thus

$$x^T y - \|x\| \leq \|x\| \left(\|y_*\| - 1 \right) \leq 0.$$

Furthermore, for $x = 0$, $x^T y - \|x\| = 0$, that is

$f^*(y) = 0$.

Summary

$$f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ \infty & \|y\|_* > 1 \end{cases}$$

Consider

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & f(x) \\ & Ax = b \\ & Cx \geq d \end{array}$$

$$L(x; \mu, \lambda) = f(x) - \mu^T (Ax - b) - \lambda^T (Cx - d)$$

$$\begin{aligned} g(\mu, \lambda) &= \inf_{x \in \operatorname{dom} f} f(x) - \mu^T (Ax - b) - \lambda^T (Cx - d) \\ &= b^T \mu + d^T \lambda + \inf_{x \in \operatorname{dom} f} f(x) - (\mu^T A + \lambda^T C) x \\ &= b^T \mu + d^T \lambda - \sup_{x \in \operatorname{dom} f} (A^T \mu + C^T \lambda)^T x - f(x) \\ &= b^T \mu + d^T \lambda - f^* (A^T \mu + C^T \lambda) \end{aligned}$$

Dual problem

$$\begin{array}{ll} \underset{\mu \in \mathbb{R}^m,\ \lambda \in \mathbb{R}^p}{\text{maximize}} & b^T \mu + d^T \lambda - f^* (A^T \mu + C^T \lambda) \\ & \lambda \geq 0 \end{array}$$

Ex

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & \|x\|_\infty \\ & Ax = b \end{array}$$

Dual problem

$$\underset{\mu \in \mathbb{R}^m}{\text{maximize}} \quad b^T\mu - \begin{cases} 0 & \text{if } \|A^T\mu\|_1 \leq 1 \\ \infty & \text{if } \|A^T\mu\|_1 > 1 \end{cases}$$

equivalently

$$\begin{array}{ll} \underset{\mu \in \mathbb{R}^m}{\text{maximize}} & b^T\mu \\ \|A^T\mu\|_1 \leq 1 & \end{array}$$