Math 450 - 558: Smooth and Nonsmooth Optimization

Instructor: Emre Mengi

Fall Semester 2015
Midterm Exam
Friday November 6th, 2015
Duration: 150 minutes

#1	20	
#2	25	
#3	30	
#4	25	
Σ	100	

Name ____________________

Student ID ____________________

Signature ____________________

- Please put your name, student ID and signature in the space provided above.
- You may use the lecture notes or your own notes, but textbooks are not allowed.

Question 1 This question concerns a quasi-Newton method based on rank two symmetric inverse Hessian updates of the form

$$H_{k+1} = H_k + u s_k^T + s_k u^T. \tag{1}$$

Above, H_{k+1} is an approximation for the inverse Hessian $\left[\nabla^2 f(x^{(k+1)})\right]^{-1}$ and $s_k := x^{(k+1)} - x^{(k)}$.

(a) (10 points) Derive an update rule of the form (1) such that H_{k+1} satisfies the secant equation.

(b) (10 points) Consider a line search method outlined below. Does this

---

Choose an estimate $x^{(0)}$ for a local minimizer.
Calculate $\nabla f(x^{(0)})$, and choose a positive definite H_0.
for $k = 0, 1, 2, 3, \ldots$ **do**
(1) $p_k \leftarrow -H_k \cdot \nabla f(x^{(k)})$
(2) Perform a line search in the direction p_k to determine a step-length α_k such that $x^{(k)} + \alpha_k p_k$ satisfies the Wolfe conditions.
(3) $x^{(k+1)} \leftarrow x^{(k)} + \alpha_k p_k$, and calculate $\nabla f(x^{(k+1)})$.
(4) Form H_{k+1} from H_k, $s_k = x^{(k+1)} - x^{(k)}$ and $y_k = \nabla f(x^{(k+1)}) - \nabla f(x^{(k)})$ using the update rule (1).
end for

---

algorithm guarantee the generation of descent search directions p_k, i.e., does p_k necessarily satisfy $f(x^{(k)} + \alpha p_k) < f(x^{(k)})$ for all $\alpha > 0$ small enough? Explain.

Question 2 (25 points) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function, and x_* is a local minimizer of f such that $\nabla^2 f(x_*)$ is *not invertible*. Furthermore, suppose that there exist a ball $B(x_*, \delta)$ with positive radius $\delta > 0$ such that $\nabla^2 f(x)$ is invertible at all $x \in B(x_*, \delta) \backslash \{x_*\}$, and that the limit

$$\lim_{x \to x_*} \left\| \left[\nabla^2 f(x) \right]^{-1} \right\|_2 \|x - x_*\|$$

exists and is finite.

Prove that there exists a ball $B(x_*, \eta)$ (with positive radius $\eta > 0$) such that, for all $x^{(0)} \in B(x_*, \eta)$, pure Newton's method for unconstrained optimization generates a sequence $\{x^{(k)}\}$ satisfying

$$\lim_{k \to \infty} x^{(k)} = x_*.$$

Question 3 Consider the inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x\in\mathbb{R}^2} & \frac{1}{2}(x_1-1)^2 - x_1 - x_2 \\ \text{subject} & x_1 - x_2^2 \geq 0 \\ & 2 - x_1^2 - x_2^2 \geq 0, \end{array} \tag{2}$$

and the points $\bar{x} = (0,0)$ and $\hat{x} = (1,1)$.

(a) (5 points) Write down a feasible sequence $\{z^{(k)}\}$ leading to $\hat{x}$. Write down also a limiting direction associated with this feasible sequence.

(b) (5 points) Find the tangent cone at $\hat{x}$.

(c) (20 points) Show, for each one of $\bar{x}$ and $\hat{x}$, either the point is a local minimizer of (2) or not a local minimizer of (2).

Question 4 Let us focus on the following linearly constrained nonlinear program.

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & f(x) \\ \text{subject} & Ax = b \\ & x \geq 0 \end{array} \tag{3}$$

Above, $A \in \mathbb{R}^{m \times n}$ is a given matrix, $b \in \mathbb{R}^m$ is a given vector, and the objective function $f(x)$ is twice continuously differentiable.

(a) (15 points) Suppose that $\nabla^2 f(x)$ is *positive definite* at all x, and let x_* be a point such that the following hold for some λ_* and $s_* \geq 0$:

$$Ax_* = b, \;\; x_* \geq 0, \;\; x_*^T s_* = 0 \;\text{ and }\; \nabla f(x_*) = A^T \lambda_* + s_*.$$

Prove that x_* is a global minimizer of (3).

(b) (10 points) Suppose that $\mathrm{Null}(A) \neq \{0\}$, and that Z is a matrix whose columns form an orthonormal basis for $\mathrm{Null}(A)$. Furthermore, suppose $Z^T \nabla^2 f(x) Z$ is *not positive semi-definite* at all x. Let x_* be a point such that the following hold for some λ_*:

$$Ax_* = b, \quad x_* > 0 \text{ and } \nabla f(x_*) = A^T \lambda_*.$$

Prove that x_* is not a local minimizer of (3).