Backward Error Analysis
(Reminders from Lecture 17)

Backward Stability

$f : V \to W$ - Problem
$\widehat{f} : V \to W$ - Computed solution by the algorithm

Definition

The algorithm is called backward stable if

$$\widehat{f}(x) \;=\; f(x + \delta x) \qquad \exists \delta x \in V \;\text{ s.t. }\; \frac{\|\delta x\|}{\|x\|} = O(\epsilon_{\mathrm{mach}})$$

for all $x \in V, x \neq 0$.

Backward Error Analysis

Suppose the algorithm is backward stable so that

$$\widehat{f}(x) \quad = \quad f(x + \widetilde{\delta x}) \qquad \exists \widetilde{\delta x} \ \text{ s.t. } \ \|\widetilde{\delta x}\| = O(\epsilon_{\text{mach}})\|x\|.$$

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Setting $\delta := \|\widetilde{\delta x}\|$, we have

$$\widetilde{E} := \frac{\|\widehat{f}(x) - f(x)\|}{\|f(x)\|} \leq \sup_{0 < \|\delta x\| \leq \delta} \frac{\|f(x + \delta x) - f(x)\|}{\|f(x)\|}$$

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$$\leq \sup_{0<\|\delta x\|\leq\delta} \frac{\|f(x+\delta x) - f(x)\|/\|f(x)\|}{\|\delta x\|/\|x\|} \cdot \frac{\|\delta x\|}{\|x\|}$$

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&\leq \widetilde{\kappa}_\delta \cdot \frac{\|\widetilde{\delta x}\|}{\|x\|}
\end{aligned}$$

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&\leq \widetilde{\kappa}_\delta \cdot \frac{\|\widetilde{\delta x}\|}{\|x\|} \;=\; \widetilde{\kappa}_\delta \cdot O(\epsilon_{\text{mach}}).
\end{aligned}$$

Backward Error Analysis

Relative Error for a Backward Stable Algorithm

$$\widetilde{E} := \frac{\|\widehat{f}(x) - f(x)\|}{\|f(x)\|} \quad \leq \quad \widetilde{\kappa}_\delta \cdot \frac{\|\widetilde{\delta x}\|}{\|x\|} = \widetilde{\kappa}_\delta \cdot O(\epsilon_{\text{mach}})$$

Example: Inner Product in $\mathbb{R}^n$

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x) = x_1 y_1 + \cdots + x_n y_n = x^T y,$$
$$\widehat{f} : \mathbb{R}^n \to \mathbb{R}, \quad \widehat{f}(x) = (x_1 \otimes y_1) \oplus \cdots \oplus (x_n \otimes y_n)$$

(for a given $y \in \mathbb{R}^n$)

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- Conditioning

$$\widetilde{\kappa}_\delta := \sup_{0<\|\delta x\|\le\delta} \frac{|(x+\delta x)^T y - x^T|/|x^T y|}{\|\delta x\|/\|x\|}$$
$$= \sup_{0<\|\delta x\|\le\delta} \frac{|\delta x^T y|\|x\|}{\|\delta x\||x^T y|} = \frac{1}{|\cos(\theta)|}$$

where θ is the angle between x and y.

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Relative Error

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