LU Factorization with Partial Pivoting

November 23, 2018

Example.

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \mapsto_{(S1)\ r_1 \leftrightarrow r_2} \begin{bmatrix} -4 & 1 & 2 \\ 1 & -2 & 1 \\ -1 & 4 & 1 \end{bmatrix} \mapsto \begin{bmatrix} -4 & 1 & 2 \\ {\color{blue}-1/4} & -7/4 & 3/2 \\ {\color{blue}1/4} & 15/4 & 1/2 \end{bmatrix}$$

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$$\mapsto_{(S2)\ r_2 \leftrightarrow r_3} \begin{bmatrix} -4 & 1 & 2 \\ \color{blue}{1/4} & 15/4 & 1/2 \\ \color{blue}{-1/4} & -7/4 & 3/2 \end{bmatrix}$$

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Permutation Matrices

$$(\text{S1}), r_1 \leftrightarrow r_2 \quad P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (\text{S2}), r_2 \leftrightarrow r_3 \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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LU Factorization Computed

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_2} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ -4 & 1 & 2 \\ -1 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} -4 & 1 & 2 \\ -1 & 4 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/4 & -7/15 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 & 2 \\ 0 & 15/4 & 1/2 \\ 0 & 0 & 26/15 \end{bmatrix}$$

Given $A \in \mathbb{R}^{n\times n}$ form $L, U \in \mathbb{R}^{n\times n}$, and p a vector of size $n-1$

$L \longleftarrow I_n$
for $k = 1, \ldots, n-1$ **do**
  Find j such that $|a_{jk}| = \max_{\ell=k,\ldots,n} |a_{\ell k}|$
  $p(k) \leftarrow j$
  % Swap rows j and k, also the multipliers
  $A(k, k:n) \longleftrightarrow A(j, k:n)$, $L(k, 1:k-1) \longleftrightarrow L(j, 1:k-1)$
  for $j = k+1, \ldots, n$ **do**
    $l_{jk} \longleftarrow a_{jk}/a_{kk}$
    $A(j, k:n) \longleftarrow A(j, k:n) - l_{jk} A(k, k:n)$
  end for
end for
$U \longleftarrow A$

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- $|l_{jk}| \leq 1$
- $p(k) = j$ means P_k is the permutation matrix permuting rows k and j.

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- Computed factorization

$$P_{n-1} \ldots P_2 P_1 A = LU$$

- \# of flops $\sim 2n^3/3$

Forward Substitution

$$
\begin{bmatrix} l_{11} & 0 & & 0 \\ l_{21} & l_{22} & & 0 \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

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$$x_1 = b_1/l_{11}$$

Forward Substitution

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$$\begin{aligned} x_1 &= b_1/l_{11} \\ x_2 &= \{b_2 - l_{21}x_1\}/l_{22} \end{aligned}$$

Forward Substitution

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$$\begin{aligned} x_1 &= b_1/l_{11} \\ x_2 &= \{b_2 - l_{21}x_1\}/l_{22} \\ x_j &= \{b_j - l_{j1}x_1 - \cdots - l_{j(j-1)}x_{j-1}\}/l_{jj} \quad j = 3, \ldots, n \end{aligned}$$

Pseudocode

for $j = 1, \ldots, n$ **do**
    $x_j \leftarrow b_j$
    for $k = 1, \ldots, j - 1$ **do**
        $x_j \leftarrow x_j - l_{jk} x_k$
    end for
    $x_j \leftarrow x_j / l_{jj}$
end for