Computation of All Eigenvalues

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A and SAS^{-1} have

- the same characteristic polynomial,
- have the same set of eigenvalues.

Unitary Similarity Transformation

For a unitary $Q \in \mathbb{C}^{n \times n}$, the transformation

$$A \mapsto QAQ^*$$

- Recall $Q^{-1} = Q^*$ for a unitary Q.

Computing All Eigenvalues, Outline

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- For an upper triangular matrix,
  the eigenvalues are the diagonal entries.