Some Details of the QR Algorithm

Choice of the Shift

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- WIlkinson Shift

  $\sigma =$ eigenvalue of $H(n-1:n, n-1:n)$ closest to $H(n, n)$

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Hence,

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Conclusion

$$\Lambda(H) = \Lambda(H_1) \cup \Lambda(H_2)$$

$\Lambda(H)$ - the set of eigenvalues (the spectrum) of H

Deflation

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Say $\Lambda(H_1) \cup \Lambda(H_2) = \{\lambda_1, \ldots, \lambda_\kappa\}$

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Say $\Lambda(H_1) \cup \Lambda(H_2) = \{\lambda_1, \ldots, \lambda_\kappa\}$

- algebraic multiplicites $r_1^{(1)}, \ldots, r_\kappa^{(1)}$ of $\lambda_1, \ldots, \lambda_\kappa$ as eigenvalues of H_1
- algebraic multiplicites $r_1^{(2)}, \ldots, r_\kappa^{(2)}$ of $\lambda_1, \ldots, \lambda_\kappa$ as eigenvalues of H_2

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$\lambda_1, \ldots, \lambda_\kappa$ are eigenvalues of H
with algebraic multiplicites $r_1^{(1)} + r_1^{(2)}, \ldots, r_\kappa^{(1)} + r_\kappa^{(2)}$.