Computation of Singular Values

Outline

Given $A \in \mathbb{C}^{m \times n}$,

1. <u>REDUCTION INTO BIDIAGONAL FORM</u>
   Form unitary $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ such that

   $$UAV = B$$

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2. <u>THE QR ALGORITHM</u>
   Form unitary $U_1, \ldots, U_k \in \mathbb{C}^{m \times m}$ and unitary $V_1, \ldots, V_k \in \mathbb{C}^{n \times n}$ such that

   $$B_k := U_k \ldots U_1 B V_1 \ldots V_k$$

   becomes diagonal as $k \to \infty$.

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   becomes diagonal as $k \to \infty$.

- $A, B, B_1, B_2, B_3, \dots$ all have the same singular values.

The QR Algorithm for Singular Values

Generates a sequence $\{B_k\}$ such that $B_0 = B$ and B_{k+1}, B_k are related as follows:

(a) Let

$$\begin{aligned} B_k^* B_k - \sigma_k I &= Q_{k+1} R_{k+1} \\ B_k B_k^* - \widetilde{\sigma}_k I &= P_{k+1} S_{k+1} \end{aligned}$$

be QR factorizations (for given shifts $\sigma_k, \widetilde{\sigma}_k$).

(b) $B_{k+1} \quad := \quad P_{k+1}^* B_k Q_{k+1}$

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2. Recalling $B_k B_k^* - \widetilde{\sigma}_k I = P_{k+1} S_{k+1}$,

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3. The multiplication

$$P_{k+1}^* B_k Q_{k+1}$$

   can be performed at a cost of $O(m+n)$ flops.