# General Minimal Residual (GMRES)

# Problem & Approach

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**GMRES**
Find $x_* \in \mathcal{K}_k \ (\subseteq \mathbb{C}^n)$ s.t.

$$\|b - Ax_*\|_2 = \min_{x \in \mathcal{K}_k} \|b - Ax\|_2$$

where $\mathcal{K}_k := \text{span}\{b, Ab, \ldots, A^{k-1}b\}$.

We hope $Ax_* \approx b$.

# Arnoldi Process

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  orthonormal basis for $\mathcal{K}_{k-1} := \text{span}\{b, Ab, \dots, A^{k-2}b\}$.
- Find $\{q_1, \dots, q_{k-1}, q_k\}$
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$$Aq_{k-1} = h_{1(k-1)}q_1 + h_{2(k-1)}q_2 + \cdots + h_{k(k-1)}q_k$$

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Letting $v := Aq_{k-1}$, due to orthogonality of $q_1, \ldots, q_k$

$$\begin{aligned} h_{j(k-1)} &= q_j^* v \quad j = 1, \ldots, k-1, \\ h_{k(k-1)} &= v - h_{1(k-1)}q_1 \cdots - h_{(k-1)(k-1)}q_{k-1}. \end{aligned}$$

# Arnoldi Process

$q_1 \leftarrow b/\|b\|_2$
**for** $j = 1, \ldots, k-1$ **do**
  $v \leftarrow Aq_j$
  **for** $\ell = 1, \ldots, j$ **do**
    $h_{\ell j} \leftarrow q_j^* v$
    $v \leftarrow v - h_{\ell j} q_j$
  **end for**
  $h_{(j+1)j} \leftarrow \|v\|_2$
  $q_j \leftarrow v/h_{(j+1)j}$
**end for**

# Arnoldi Recurrence

$$A \underbrace{\begin{bmatrix} q_1 & q_2 & \dots & q_{k-1} \end{bmatrix}}_{Q_{k-1}} =$$

$$\underbrace{\begin{bmatrix} q_1 & q_2 & \dots & q_{k-1} & q_k \end{bmatrix}}_{Q_k} \underbrace{\begin{bmatrix} h_{11} & h_{12} & \dots & & h_{1(k-1)} \\ h_{21} & h_{22} & & & h_{2(k-1)} \\ 0 & h_{32} & \ddots & & h_{3(k-1)} \\ 0 & 0 & \ddots & \ddots & \\ \vdots & & & \ddots & h_{kk} \\ 0 & 0 & & & h_{(k+1)k} \end{bmatrix}}_{H_k}$$

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- $Q_k \in \mathbb{C}^{n \times k}$ columns form orthonormal basis for $\mathcal{K}_k$
- $H_k \in \mathbb{C}^{(k+1) \times k}$ Hessenberg

# The Least Squares Problem

$$\min_{x \in \mathcal{K}_k} \|b - Ax\|_2 \quad = \quad \min_{\alpha \in \mathbb{C}^k} \|b - AQ_k\alpha\|_2$$

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&= \min_{\alpha \in \mathbb{C}^k} \|b - Q_{k+1} H_{k+1} \alpha\|_2 \\
&= \min_{\alpha \in \mathbb{C}^k} \|Q_{k+1}^* b - H_{k+1} \alpha\|_2
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\end{aligned}$$

**Procedure**

1. Form $Q_{k+1}$ and $H_{k+1}$ by the Arnoldi Procedure.
2. Find $\alpha_k \in \mathbb{C}^k$ s.t.

$$\|Q_{k+1}^* b - H_{k+1}\alpha_k\|_2 = \min_{\alpha\in\mathbb{C}^k} \|Q_{k+1}^* b - H_{k+1}\alpha\|_2$$

3. $x_* := Q_k\alpha_k$ is the (approximate) solution.