Minimization Property of GMRES & Preconditioning

The Minimization Property

GMRES

Find $x^{(k)} \in \mathcal{K}_k \ (\subseteq \mathbb{C}^n)$ s.t.

$$\|b - Ax^{(k)}\|_2 = \min_{x \in \mathcal{K}_k} \|b - Ax\|_2$$

where $\mathcal{K}_k := \mathrm{span}\{b, Ab, \ldots, A^{k-1}b\}$.

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Theorem

$$\|b - Ax^{(k)}\|_2 \;\le\; \|p(A)b\|_2 \quad \forall p \in \mathcal{P}_k^0$$

where

$$\mathcal{P}_k^0 := \Big\{p : \mathbb{C} \to \mathbb{C},\ p(z) = 1 + \alpha_1 z + \cdots + \alpha_k z^k \;\Big|\; \alpha_1, \ldots, \alpha_k \in \mathbb{C}\Big\}.$$

Corollaries

Let $A \in \mathbb{C}^{n \times n}$ have n linearly independent eigenvectors with the eigenvalue decomposition $A = V \Lambda V^{-1}$.

For every $p \in \mathcal{P}_k^0$, we have

$$\|b - Ax^{(k)}\|_2 \;\leq\; \|V\|_2 \|V^{-1}\|_2 \|b\|_2 \left\{ \max_{z \in \Lambda(A)} |p(z)| \right\}$$

Corollaries

In particular, if A has k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, setting $p(z) := \prod_{j=1}^{k} (\lambda_j - z)/\lambda_j \in \mathcal{P}_k^0$,

$$\max_{z \in \Lambda(A)} |p(z)| = 0 \implies \|b - Ax^{(k)}\|_2 = 0.$$

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Theorem

Suppose $A \in \mathbb{C}^{n \times n}$ has k distinct eigenvalues. Then $Ax^{(k)} = b$.

Preconditioning

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Apply GMRES to $MAx = Mb$.
Letting $\widehat{x}^{(k)}$ be the best solution in $\mathcal{K}_k$,

$$\begin{aligned} \|Mb - MA\widehat{x}^{(k)}\|_2 &\leq \|p(MA)(Mb)\|_2 \\ &\leq \|p(MA)\|_2 \|Mb\|_2 \end{aligned}$$

for any $p \in \mathcal{P}_k^0$.

Preconditioning

Choosing $p(z) := (1 - z)^k \in \mathcal{P}_k^0$, we obtain

$$\|Mb - MA\widehat{x}^{(k)}\|_2 \quad \leq \quad \|I - MA\|_2^k \cdot \|Mb\|_2$$

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- but M has to be cheap to compute.

Preconditioning

Some simple choices when A is diagonally dominant

(1) $M = D^{-1}$

(2) $M = (L + D)^{-1}$

where D, L are diagonal, lower triangular parts of A.

Polynomial Preconditioning

Choose $M = P(A)$ so that

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- Choose $p(z)$ s.t. $p(\lambda_j) \approx 1/\lambda_j$ for $j = 1, \ldots, n$.