Summary of Lecture 1

September 19, 2018

Definition 0.1 (Inner Product). *An inner product on a real (complex) vector space is a function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \quad$ *(* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$*) satisfying the following properties:*

(i) $\langle v, v \rangle$ *is real and positive* $\quad \forall v \in V,\ v \neq 0.$

(ii) $\langle v, w \rangle = \langle w, v \rangle$ *(*$\langle v, w \rangle = \overline{\langle w, v \rangle}$*)* $\quad \forall v, w \in V$*.*

(iii) $\langle v, \alpha w \rangle = |\alpha| \langle v, w \rangle \quad \forall v, w \in V$*,* $\forall \alpha \in \mathbb{R} \quad$ *(*$\forall v, w \in V$*,* $\forall \alpha \in \mathbb{C}$*).*

(iv) $\langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle \quad \forall v, u, w \in V$*.*

Standard Inner Products.

$\mathbb{R}^n$ - $\quad \langle x, y \rangle = x_1 y_1 + \ldots, x_n y_n = x^T y$

$\mathbb{C}^n$ - $\quad \langle x, y \rangle = \overline{x}_1 y_1 + \ldots, \overline{x}_n y_n = x^* y$

$\mathbb{R}^{m \times n}$ - $\quad \langle A, B \rangle = \text{Trace}(A^T B) = \sum_{j=1}^m \sum_{k=1}^n a_{jk} b_{jk}$

Euclidean Norms

V a vector space with an inner product $\langle \cdot, \cdot \rangle$

$$\|v\| := \sqrt{\langle v, v \rangle} \quad \forall v \in V$$

Standard Euclidean Norms.

$\mathbb{R}^n$ - $\quad \|x\| = \sqrt{x^T x} = \sqrt{x_1^2 + \cdots + x_n^2}$

$\mathbb{C}^n$ - $\quad \|x\| = \sqrt{x^* x} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$

$\mathbb{R}^{m \times n}$ - $\quad \|A\| = \sqrt{\mathrm{Trace}(A^T A)} = \sqrt{\sum_{j=1}^m \sum_{k=1}^n a_{jk}^2}$

Definition 0.2 (Norm)**.** *A norm on a vector space* V *is a function* $\|\cdot\| : V \to \mathbb{R}$ *satisfying*

(i) $\|v\| > 0 \quad \forall v \in V, v \neq 0$,

(ii) $\|\alpha v\| = |\alpha|\|v\| \quad \forall v \in V, \forall \alpha \in \mathbb{R} \quad (\forall v \in V, \forall \alpha \in \mathbb{C})$,

(iii) $\|v + w\| \leq \|v\| + \|w\|$.

Common Norms in $\mathbb{R}^n$ or $\mathbb{C}^n$

1-norm $\quad \|x\|_1 := |x_1| + \cdots + |x_n|$

∞-norm $\quad \|x\|_\infty := \max_{j=1,\ldots,n} |x_j|$

p-norm $\quad \|x\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$

Equivalence of Norms.
$\exists C_1, C_2 \quad \text{s.t} \quad C_1\|v\|_B \leq \|v\|_A \leq C_2\|v\|_B \quad \forall v \in V$