MATH 504: Numerical Methods - I

Final - Fall 2010

Duration : 180 minutes

#1	20	
#2	15	
#3	20	
#4	15	
#5	15	
#6	15	
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NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1. Consider the matrices

$$A_1 = \begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 2 & 4 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find the eigenvalues of A_1 and the eigenspace associated with each of its eigenvalues.

(b) Find the eigenvalues of A_2 together with their algebraic and geometric multiplicities.

(c) Find a Schur factorization for A_1.

(d) Let v_0 and v_1 be two linearly independent eigenvectors of A_1. Suppose also that $\{q_k\}$ denotes the sequence of vectors generated by the inverse iteration with shift $\sigma = 2$ and starting with an initial vector $q_0 = \alpha_0 v_0 + \alpha_1 v_1 \in \mathbb{C}^2$ where α_0, α_1 are nonzero scalars.

Determine the subspace that $\text{span}\{q_k\}$ is approaching as $k \to \infty$.

Question 2. Suppose that you are given a full singular value decomposition for $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ of the form

$$A = U\Sigma V^* \tag{1}$$

where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{C}^{m\times n}$ is diagonal with positive entries along the diagonal.

(a) First assume that $m = n$ so that A is a square matrix. Given the SVD in (1) of A. Describe an algorithm to solve the linear system $Ax = b$ at a cost of $O(n^2)$ where $b \in \mathbb{C}^n$.

(b) Now assume that $m > n$. Given the SVD in (1) of A. Describe an algorithm to solve the least squares problem

$$\text{find } x \in \mathbb{C}^n \text{ so that } \|Ax - b\|_2 \text{ is as small as possible}$$

at a cost of $O(m^2)$ where $b \in \mathbb{C}^m$.

Question 3. This question concerns the sensitivity of numerical problems and stability of numerical algorithms.

(a) Which of the following operations can be performed in a backward stable manner in IEEE floating point arithmetic? Explain your reasoning.

(i) The summation $f(x) = 3 + x$ as a function of $x \in \mathbb{R}$.

(ii) The scalar multiplication $g(x) = 3 \cdot x$ as a function of $x \in \mathbb{R}$.

(b) Given a fixed $b \in \mathbb{R}^2$, and the matrices

$$A_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 1 \\ -10^{-3} & 10^{-3} \end{bmatrix}.$$

Which of the linear systems $A_1 x = b$ and $A_2 x = b$ is more sensitive to perturbations in A_1 and A_2, respectively? Explain.

(c) Given the vector

$$b = \begin{bmatrix} -1 \\ 10^{-3} \\ 1 \end{bmatrix}$$

and the matrices

$$\tilde{A}_1 = \begin{bmatrix} 3 & 1 \\ 1 & -2 \\ 3 & 1 \end{bmatrix} \text{ and } \tilde{A}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix}.$$

Let

– x_1 be the solution of the least squares problem

$$\text{minimize}_x \|A_1 x - b\|_2, \tag{2}$$

– x_2 be the solution of the least squares problem

$$\text{minimize}_x \|A_2 x - b\|_2 \tag{3}$$

Is the solution x_1 of problem (2) w.r.t. perturbations in A_1 more sensitive or the solution x_2 of problem (3) w.r.t. perturbations in A_2? Explain.

Question 4. This question concerns the LU factorization of tridiagonal matrices without pivoting.

(a) Find an LU factorization for the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

(b) Suppose $A \in \mathbb{C}^{n\times n}$ is tridiagonal with $a_{ij} = 0$ whenever $|i-j| > 1$. Suppose also that A is reducible into a triangular matrix by only applying row-replacement operations. Show that A has an LU factorization of the form

$$A = \underbrace{\begin{bmatrix} 1 & 0 & \dots & & 0 \\ \ell_{21} & 1 & & & 0 \\ 0 & \ell_{32} & 1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \ell_{n(n-1)} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & 0 & \dots & 0 \\ 0 & u_{22} & u_{23} & & 0 \\ 0 & & u_{33} & \ddots & \vdots \\ \vdots & & & \ddots & u_{(n-1)n} \\ 0 & \dots & & 0 & u_{nn} \end{bmatrix}}_{U} \tag{4}$$

that is A can be written of the form $A = LU$ where

- L is lower triangular with the entries along the diagonal equal to one, the entries along the subdiagonal possibly non-zero and all other entries equal to zero,
- U is upper triangular with the entries on the diagonal and the superdiagonal non-zero, and all other entries equal to zero.

(c) Write a pseudocode that requires $O(n)$ flops for the calculation of the LU factorization of form (4) for a tridiagonal matrix A.

Question 5. Given an upper triangular non-singular matrix $R \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$. Suppose that the linear system

$$R^4\, x = b \tag{5}$$

is solved by performing back-substitution four times. In particular let $x_0 = b$. Then the system $Rx_i = x_{i-1}$ is solved for $i = 1, \dots, 4$ by back substitution. The solution to (5) is given by $x = x_4$. Below in part **(a)** you can refer to the following result.

<u>Theorem:</u>
Given a non-singular upper triangular matrix $R \in \mathbb{C}^{n\times n}$ and $c \in \mathbb{C}^n$. Suppose that the linear system $Ry = c$ is solved by back substitution in IEEE floating point arithmetic. Then the computed solution $\hat{y}$ satisfies

$$(R + \widehat{\delta R})\hat{y} = c$$

for some $\widehat{\delta R}$ such that

$$\frac{\|\widehat{\delta R}\|_1}{\|R\|_1} \leq n\epsilon_{\text{mach}} + O(\epsilon_{\text{mach}}^2)$$

(a) Show that the computed solution $\hat{x}$ for the system (5) satisfies

$$(R + \delta R)^4 \, \hat{x} = b$$

for some δR. Find a tight upper bound for the relative backward error

$$\frac{\|\delta R\|_1}{\|R\|_1}.$$

(b) Find a tight upper bound for the relative forward error

$$\frac{\|\hat{x} - x\|_1}{\|x\|_1}$$

where $\hat{x}$ is the computed solution as defined in part **(a)**.

Question 6. The QR algorithm is one of the standard approaches to compute the eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$. In this question you are expected to shed a light into the relation between the QR algorithm and simultaneous iteration. Pseudocodes are provided below for the QR algorithm as well as for the simultaneous iteration.

Algorithm 1 The QR Algorithm

```
A_0 ← A
for k = 0, 1, ... do
    Compute a QR factorization A_k = Q_{k+1} R_{k+1}
    A_{k+1} ← R_{k+1} Q_{k+1}
end for
```

Algorithm 2 Simultaneous Iteration

```
for k = 1, ... do
    Compute a QR factorization A^k = Q̂_k R̂_k
    Â_k ← Q̂_k^* A Q̂_k
end for
```

Show that a QR factorization for A^k is given by

$$A^k = \underbrace{Q_1 Q_2 \ldots Q_k}_{\hat{Q}_k} \underbrace{R_k \ldots R_2 R_1}_{\hat{R}_k}.$$

(Hint: First try to express A_k in terms of A and Q_j for $j = 1, \ldots, k$.)