Math 504: Numerical Methods - I

Final - Fall 2012
Duration : 180 minutes

Name ____________________

Student ID ____________________

Signature ____________________

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- Put your name and student ID in the space provided above.
- Pick any six out of the seven questions; circle the questions that you selected.
- The exam is out of 120 points; there is no bonus.
- No calculators or any other electronic devices are allowed.
- This is a closed-book exam, but you can use notes.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1. Let $A \in \mathbb{C}^{n\times n}$ and $z \in \mathbb{C}^n$. Show that

(i) $\inf\{\|\delta A\|_2 \mid \delta A \in \mathbb{C}^{n\times n} \text{ s.t. } z \text{ is an eigenvalue of } (A+\delta A)\} = \sigma_n(A - zI)$, where $\sigma_n(A-zI)$ denotes the smallest singular value of $A - zI$.

(ii) Furthermore, a minimal perturbation solving the minimization problem in **(i)** is given by $\delta A_* = -\sigma u_n v_n^*$ where $\sigma := \sigma_n(A - zI)$ and u_n, v_n denotes a pair of unit left and right singular vectors associated with $\sigma_n(A - zI)$.

Question 2. For a given matrix $A \in \mathbb{C}^{m\times n}$ with $m > n$, find a unitary matrix $Q \in \mathbb{C}^{m\times m}$ such that Qv is the reflection of v about $\mathrm{Col}(A)$ for each $v \in \mathbb{C}^m$.

Question 3. Write down a pseudocode to compute the QR factorization of a matrix $A \in \mathbb{C}^{m\times n}$ by using Givens' rotators. Perform also a flop count for your pseudocode.

Question 4. This question concerns a Hermitian matrix $A \in \mathbb{C}^{n\times n}$. Let λ be an eigenvalue, $v \in \mathbb{C}^n$ be an associated unit eigenvector of A, and $r(q) := q^*Aq$ denote the Rayleigh quotient associated with the unit vector $q \in \mathbb{C}^n$. Show that

$$|\lambda - r(q)| \leq \kappa \|v - q\|_2^2$$

where $\kappa := \max_{k=2,\ldots,n} |\lambda - \lambda_k|$ with $\lambda, \lambda_2, \ldots, \lambda_n$ representing the set of (not necessarily distinct) eigenvalues of A.

Question 5. Consider the shifted version of the QR algorithm, for which a pseudocode is given below.

Algorithm 1 The QR Algorithm with Shifts

```
A_0 ← A
for k = 0, 1, ... do
    Choose a shift μ_k
    Compute a QR factorization A_k − μ_k I = Q_{k+1} R_{k+1}
    A_{k+1} ← R_{k+1} Q_{k+1} + μ_k I
end for
```

Show that the computed orthogonal factors Q_j and upper triangular factors R_j satisfy

$$(A - \mu_k I)\ldots(A - \mu_0 I) = Q_1 \ldots Q_{k+1} R_{k+1} \ldots R_1.$$

(Note: This means that the shifted QR algorithm mimics a simultaneous Rayleigh iteration.)

Question 6. For given $A \in \mathbb{C}^{n\times n}$ and $q_0 \in \mathbb{C}^n$, suppose that a sequence of vectors $q_1, \ldots, q_k$ related by $Aq_{j-1} = q_j,\ j = 1, \ldots, k$ is generated by applying the matrix vector product repeatedly in a computer satisfying IEEE floating point standards. Let $\tilde{q}_1, \ldots, \tilde{q}_k$ denote the computed vectors. Perform a backward error analysis to deduce a tight upper bound for the forward error

$$\frac{\|\tilde{q}_k - q_k\|_1}{\|q_k\|_1}.$$

Question 7. Consider the tridiagonal symmetric matrix $A \in \mathbb{R}^{5\times 5}$ with $a_{11} = 3$, $a_{jj} = 10/3$ for $j = 2, \ldots, 5$, $a_{j+1,j} = a_{j,j+1} = 1$ for $j = 1, \ldots, 4$ and all other entries equal to zero.

(a) Show that A is positive definite.

(b) Find a Cholesky factorization of A.