Math 504, Fall 2018 - Homework 2

October 27, 2018

1. Let

$$\mathcal{S}_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{S}_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Find the projector onto $\mathcal{S}_1$ along $\mathcal{S}_2$.

(b) Find the orthogonal projector onto $\mathcal{S}_1$.

2. Consider a matrix $A \in \mathbb{C}^{m \times n}$ with the full SVD

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}$$

where $\Sigma_1 \in \mathbb{R}^{r \times r}$ is diagonal with positive diagonal entries, $U_1 \in \mathbb{C}^{m \times r}$, $U_2 \in \mathbb{C}^{m \times (m-r)}$, $V_1 \in \mathbb{C}^{n \times r}$ and $V_2 \in \mathbb{C}^{n \times (n-r)}$.

Write down expressions for the orthogonal projectors onto $\mathrm{Col}(A)$, $\mathrm{Col}(A)^\perp$, $\mathrm{Null}(A)$, $\mathrm{Null}(A)^\perp$ in terms of U_1, U_2, V_1, V_2.

3. Suppose that $P \in \mathbb{C}^{n \times n}$ is a projector. Show $I - P$ is also a projector onto Null(P) along Col(P).

4. Let $F \in \mathbb{R}^{m \times m}$ be the matrix such that

$$F \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_m \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 + x_m \\ x_2 + x_{m-1} \\ x_3 + x_{m-2} \\ \vdots \\ x_m + x_1 \end{bmatrix}$$

and m is even. Is F a projector? If it is a projector, is it an orthogonal projector? If it is an orthogonal projector, find an orthogonal projector onto $\mathrm{Null}(F)$.

5. In the class, we have discussed about the solution of a linear system

$$Ax = b \tag{0.1}$$

for a given $A \in \mathbb{C}^{n \times n}$, $b \in \mathbb{C}^n$, and when n is very large.

Assuming the column space of A and b, as well as the solution x lie in a small dimensional subspace $\mathcal{V}$ of $\mathbb{C}^n$, the linear system can be approximated by

$$VV^*AVV^*x \approx VV^*b,$$

where the columns of V form an orthonormal basis for $\mathcal{V}$. Hence, we may as well solve

$$V^*AVy = V^*b. \tag{0.2}$$

Then the solutions of (0.1) and (0.2) are related by $x \approx Vy$.

Implement a Matlab routine that solves the projected linear system (0.2) rather than the original linear system (0.1) with the columns of V forming an orthonormal basis for the Krylov subspace

$$\mathcal{K}_r(A, b) := \text{span}\{b, Ab, A^2b, \ldots, A^{r-1}b\}.$$

Your routine should proceed with Krylov subspaces of increasing dimension recalling that the solution x of the original system (0.1) is approximated by Vy. It should terminate when the approximate solutions with two consecutive Krylov subspaces differ by less than a prescribed tolerance.

Test your Matlab routine with two particular linear systems provided together with this homework. In each case, check also $\|A\widetilde{x} - b\|_2$ for the computed approximate solution $\widetilde{x} = Vy$.

6. Compute full QR factorizations of A given below by a Givens rotator and B below by a Householder reflector.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Perform all calculations by hand.

7. Find a unitary matrix $Q \in \mathbb{R}^{5\times5}$ such that

$$Q \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

8. A matrix S is called tridiagonal if $s_{ij} = 0$ whenever $|i - j| > 1$. For instance the matrix given below is tridiagonal.

$$\begin{bmatrix} 4 & 3 & 0 & 0 \\ -2 & 1 & -5 & 0 \\ 0 & -3 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

Devise an algorithm, in particular write down a pseudocode, to compute a factorization of a given matrix $A \in \mathbb{C}^{m\times n}$ of the form

$$A \;=\; USV^*$$

where $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$ are unitary, and $S \in \mathbb{C}^{m\times n}$ is tridiagonal. Provide also the number of flops required by your algorithm.

9. Implement a Matlab routine to compute a full QR factorization of $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ using Givens rotators. Make sure that the number of flops required by your algorithm is as few as possible. You do not need to form the Q factor explicitly, rather you could return the Givens rotators defining Q.

The next three questions are not the part of the homework. Your solutions to these will not be evaluated.

10. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of $\mathbb{C}^n$ such that $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathbb{C}^n$. Furthermore, suppose a basis $\{q_1, \dots, q_k\}$ for $\mathcal{S}_1$ and a basis $\{\widetilde{q}_1, \dots, \widetilde{q}_{n-k}\}$ for $\mathcal{S}_2$ are given. Write down the projector onto $\mathcal{S}_1$ along $\mathcal{S}_2$ in terms of $q_1, \dots, q_k, \widetilde{q}_1, \dots, \widetilde{q}_{n-k}$.

11. Let $\mathcal{S}$ be an n dimensional subspace of $\mathbb{C}^m$ where $m > n$ with an orthonormal basis $\{q_1, q_2, \dots, q_n\}$. Determine an expression for the reflector $Q \in \mathbb{C}^{m\times m}$ that reflects about $\mathcal{S}$ in terms of $q_1, q_2, \dots, q_n$.

12. A matrix H is called Hessenberg if $h_{ij} = 0$ whenever $i - j > 1$. For instance the matrix given below is Hessenberg.

$$\begin{bmatrix} 4 & 3 & 2 & -1 \\ -2 & 1 & 3 & 2 \\ 0 & -4 & 1 & 3 \\ 0 & 0 & 5 & 4 \end{bmatrix}$$

Devise an efficient algorithm to compute a full QR factorization of a given Hessenberg matrix $H \in \mathbb{C}^{m\times n}$ with $m \geq n$ based on the Householder reflectors. Make sure that the number of flops required by your algorithm is $O(n^2)$. You do not need to form the Q factor explicitly, you could instead return the reflection vectors defining Q.