W10

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Computation of One Eigenvalue

Given $A \in \mathbb{C}^{n\times n}$, suppose

$$\underbrace{A}_{n\times n}\underbrace{v}_{n\times 1} = \underbrace{\lambda}_{\text{scalar}} v \qquad \exists v \neq 0$$

$\lambda \in \mathbb{C}$ - an eigenvalue

$v \in \mathbb{C}^n$ - an eigenvector corresponding to λ

Power Iteration

Generates a sequence $\{q^{(k)}\}$ such that

$$q^{(k+1)} = A q^{(k)} / \| A q^{(k)} \|$$

for a given $q^{(0)} \in \mathbb{C}^n$.

Order the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A so that

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

$v^{(j)}$ is the $\underset{\text{unit}}{\downarrow}$ eigenvector corresponding to λ_j $j = 1, \ldots, n$

Assumptions

(1) $\{v^{(1)}, \dots, v^{(n)}\}$ is linearly independent.

(2) $|\lambda_1| > |\lambda_2|$

Expand $q^{(0)}$ in terms of eigenvectors

$$q^{(0)} = \alpha_1 v^{(1)} + \alpha_2 v^{(2)} + \dots + \alpha_n v^{(n)}$$

Assumption (3) – $\alpha_1 \neq 0$

(But think also what happens to the arguments below if $\alpha_1 = 0$.)

- $$q^{(1)} = \frac{A q^{(0)}}{\|A q^{(0)}\|_2} \qquad \left(\text{let } s_1 := \frac{1}{\|A q^{(0)}\|_2} \right)$$

$$= s_1 \{\alpha_1 A v^{(1)} + \alpha_2 A v^{(2)} + \dots + \alpha_n A v^{(n)}\}$$

$$= s_1 \alpha_1 \lambda_1 v^{(1)} + s_1 \alpha_2 \lambda_2 v^{(2)} + \dots + s_1 \alpha_n \lambda_n v^{(n)}$$

$$q^{(2)} = \frac{A q^{(1)}}{\|A q^{(1)}\|_2} = \frac{A^2 q^{(0)}}{\|A^2 q^{(0)}\|_2} \qquad \left(\text{let } s_2 := \frac{1}{\|A^2 q^{(0)}\|_2} \right)$$

$$= s_2 \alpha_1 \lambda_1^2 v^{(1)} + s_2 \alpha_2 \lambda_2^2 v^{(2)} + \dots + s_2 \alpha_n \lambda_n^2 v^{(n)}$$

$\vdots$

$$q^{(k)} = \frac{A q^{(k-1)}}{\|A q^{(k-1)}\|_2} = \frac{A^k q^{(0)}}{\|A^k q^{(0)}\|_2} \qquad \left(\text{let } s_k := \frac{1}{\|A^k q^{(0)}\|_2} \right)$$

$$= s_k \alpha_1 \lambda_1^k v^{(1)} + s_k \alpha_2 \lambda_2^k v^{(2)} + \dots + s_k \alpha_n \lambda_n^k v^{(n)}$$

$$= s_k \lambda_1^k \left\{ \alpha_1 v^{(1)} + \alpha_2 \underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^k}_{\to 0 \text{ as } k \to \infty} v^{(2)} + \dots + \alpha_n \underbrace{\left(\frac{\lambda_n}{\lambda_1}\right)^k}_{\to 0 \text{ as } k \to \infty} v^{(n)} \right\}$$

Observations

(1) $|s_k \lambda_1^k|$ is bounded from above $\forall k$

(2) $\lim_{k \to \infty} |s_k \lambda_1^k \alpha_1| = 1$

(3) $\lim_{k \to \infty} \| q^{(k)} - s_k \lambda_1^k \alpha_1 v_1 \|_2$

$$= \lim_{k \to \infty} \left\| s_k \lambda_1^k \left\{ \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v^{(2)} + \cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v^{(n)} \right\} \right\|_2$$

$$= 0.$$

Conclusion

$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\{v^{(1)}\}$

as $k \to \infty$

By this we mean

there exists a sequence $\{w_k\}$ of scalars such that $|w_k| \geq M$ $\forall k$ large enough for some $M > 0$ and

$$\lim_{k \to \infty} \| q^{(k)} - w_k v^{(1)} \|_2 = 0$$

Ex

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \qquad \lambda_1 = 6 \quad \lambda_2 = 2$$

$$v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Power iteration sequence $\{q^{(k)}\}$ satisfies

$$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \quad \text{as } k \to \infty$$

(provided $q^{(0)}$ is not a multiple of $v^{(2)}$)

Rate-of-convergence

Suppose $\lim_{k \to \infty} q^{(k)} = q_*$

A linear convergence occurs if

$$\lim_{k \to \infty} \frac{\| q^{(k+1)} - q_* \|_2}{\| q^{(k)} - q_* \|_2} = \mu$$

for some $\mu \in (0, 1)$. e.g. $\left\{\frac{1}{2^k}\right\}$ converges to 0 linearly

A superlinear convergence occurs if

$$\lim_{k \to \infty} \frac{\| q^{(k+1)} - q_* \|_2}{\| q^{(k)} - q_* \|_2} = 0$$

e.g. $\left\{\frac{1}{k!}\right\}$ converges to 0 superlinearly.

A quadratic convergence occurs if

$$\lim_{k\to\infty} \frac{\| q^{(k+1)} - q_* \|_2}{\| q^{(k)} - q_* \|_2^2} = M$$

for some $M > 0$. e.g. $\left\{ \frac{1}{10^{2^k}} \right\}$ converges to 0 quadratically

A convergence with order $p > 1$ occurs if

$$\lim_{k\to\infty} \frac{\| q^{(k+1)} - q_* \|_2}{\| q^{(k)} - q_* \|_2^p} = M$$

for some $M > 0$. e.g: $\left\{ \frac{1}{10^{p^k}} \right\}$ converges to 0 with order p.

Regarding power iteration (Assume further (4) $|\lambda_2| > |\lambda_3|$ (5) $\alpha_2 \neq 0$

$$\lim_{k\to\infty} \frac{\| q^{(k+1)} - s_{k+1} \lambda_1^{k+1} \alpha_1 v^{(1)} \|_2}{\| q^{(k)} - s_k \lambda_1^k \alpha_1 v^{(1)} \|_2}$$

$$= \lim_{k\to\infty} \frac{\| \overbrace{s_{k+1} \alpha_1 \lambda_1^{k+1}}^{\to 1 \text{ in } |\cdot| \text{ as } k\to\infty} \left\{ \frac{\alpha_2}{\alpha_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k+1} v^{(2)} + \cdots + \frac{\alpha_n}{\alpha_1}\left(\frac{\lambda_n}{\lambda_1}\right)^{k+1} v^{(n)} \right\} \|_2}{\| \underbrace{s_k \alpha_1 \lambda_1^k}_{\to 1 \text{ in } |\cdot| \text{ as } k\to\infty} \left\{ \frac{\alpha_2}{\alpha_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{k} v^{(2)} + \cdots + \frac{\alpha_n}{\alpha_1}\left(\frac{\lambda_n}{\lambda_1}\right)^{k} v^{(n)} \right\} \|_2}$$

$$= \left| \frac{\lambda_2}{\lambda_1} \right|$$

divide both the numerator & denominator by $\left| \frac{\alpha_2}{\alpha_1} \left(\frac{\lambda_2}{\lambda_1}\right)^k \right|$

Ex

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \qquad \begin{array}{ll} \lambda_1 = 7 & \lambda_2 = 3 \\ v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} & v^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}$$

$$\mathrm{span}\{q^{(k)}\} \longrightarrow \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\lim_{k \to \infty} \frac{\| q^{(k+1)} - s_{k+1} \lambda_1^{k+1} \alpha_1 v^{(1)} \|_2}{\| q^{(k)} - s_k \lambda_1^k \alpha_1 v^{(1)} \|_2} = 3/7$$

(provided $q^{(0)}$ is not a multiple of $v^{(2)}$)

Inverse Iteration

Generates a sequence $\{q^{(k)}\}$ such that

$$q^{(k+1)} = \frac{(A - \sigma I)^{-1} q^{(k)}}{\| (A - \sigma I)^{-1} q^{(k)} \|_2}$$

for a given $q^{(0)} \in \mathbb{C}^n$ and $\sigma \in \mathbb{C}$.

Observe

$$Av = \lambda v \iff (A - \sigma I) v = (\lambda - \sigma) v$$

$$\iff (A - \sigma I)^{-1} v = (\lambda - \sigma)^{-1} v$$

Eigenvalues & eigenvectors of A

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

$$v^{(1)}, v^{(2)}, \ldots, v^{(n)}$$

Eigenvalues & eigenvectors of $(A - \sigma I)^{-1}$

$$(\lambda_1 - \sigma)^{-1}, (\lambda_2 - \sigma)^{-1}, \ldots, (\lambda_n - \sigma)^{-1}$$

$$v^{(1)}, v^{(2)}, \ldots, v^{(n)}$$

λ_j is the eigenvalue of A closest to σ

$$\Longleftrightarrow$$

$(\lambda_j - \sigma)^{-1}$ is the largest eigenvalue of $(A - \sigma I)^{-1}$ in $|\cdot|$

Consequently,

$$\operatorname{span}\{q^{(k)}\} \longrightarrow \operatorname{span}\{v^{(j)}\}$$

as $k \to \infty$.

Rate-of-Convergence

Suppose λ_ℓ is the eigenvalue of A second closest to σ.

$$\lim_{k \to \infty} \frac{\| q^{(k+1)} - s_{k+1} \alpha_j (\lambda_j - \sigma)^{-(k+1)} v^{(j)} \|_2}{\| q^{(k)} - s_k \alpha_j (\lambda_j - \sigma)^{-k} v^{(j)} \|_2}$$

$$= \left| \frac{(\lambda_\ell - \sigma)^{-1}}{(\lambda_j - \sigma)^{-1}} \right| = \left| \frac{\lambda_j - \sigma}{\lambda_\ell - \sigma} \right|$$

Ex

Inverse iteration applied to

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix} \qquad \lambda_1 = 6 \quad \lambda_2 = 2$$

$$v^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

with $\sigma = 0$ satisfies

$$\text{span}\{q^{(k)}\} \longrightarrow \text{span}\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\} \quad \text{as } k \to \infty$$

(provided $q^{(0)}$ is not a multiple of $v^{(1)}$)

Same is true with $\sigma = 1$. But

$$\frac{|\lambda_2 - 1|}{|\lambda_1 - 1|} = \frac{1}{5} \quad \text{and} \quad \frac{|\lambda_2 - 0|}{|\lambda_1 - 0|} = \frac{1}{3},$$

convergence is faster with $\sigma = 1$.

Pseudocode for inverse iteration

Compute an LU factor. with partial pivoting

$$P(A - \sigma I) = LU$$

for $k = 0, 1, 2, 3, \ldots$

$\quad \hat{q} \leftarrow P q^{(k)}$

$\quad$ Solve $Ly = \hat{q}$ for y by forward substitution

$\quad$ Solve $Ux = y$ for x by back substitution

$\quad q^{(k+1)} \leftarrow x / \|x\|_2$

end

* Every iteration requires $O(n^2)$ flops.

Rayleigh - quotient

$q \in \mathbb{C}^n$ - an estimate for an eigenvector of A

An estimate for the corresponding eigenvalue:
minimizer of

$$(+) \quad \min_{\lambda \in \mathbb{C}} \| \underbrace{Aq}_{\hat{b}} - \underset{\hat{x}}{\lambda} \underset{\hat{A}}{q} \|_2$$

(+) is an LSP. Its minimizer λ_* satisfies

$$\hat{A} \lambda_* = \text{orthogonal projection of } \hat{b} \text{ onto } \mathrm{Col}(\hat{A})$$

$$q \lambda_* = \frac{q q^*}{\|q\|_2^2} (Aq)$$

$$\Longrightarrow \quad \lambda_* = q^* A q / q^* q$$

Rayleigh quotient

$$r : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}$$

$$r(q) := q^* A q / q^* q$$

THM (Error of Rayleigh-quotient)

Let $q \in \mathbb{C}^n$ be a unit vector, λ and v be an eigenvalue and a corresponding unit eigenvector of $A \in \mathbb{C}^{n \times n}$. Then

$$|r(q) - \lambda| \leqslant 2 \|A\|_2 \|q - v\|_2 .$$

Proof

$$\begin{aligned}
|r(q) - \lambda| &= |q^* A q - v^* A v| \\
&= |q^* A (q-v) - (v-q)^* A v| \\
&\leqslant |q^* A (q-v)| + |(v-q)^* A v| \\
&\leqslant 2 \|A\|_2 \|q-v\|_2 .
\end{aligned}$$

$\square$

Rayleigh quotient iteration

Generates a sequence $\{q^{(k)}\}$ such that

$$q^{(k+1)} = \frac{(A - \mu_k I)^{-1} q^{(k)}}{\|(A - \mu_k I)^{-1} q^{(k)}\|_2}$$

where

$$\mu_k = (q^{(k)})^* A \, q^{(k)}$$

Suppose $q^{(k)} \to v^{(j)}$ as $k \to \infty$, where $v^{(j)}$ is a unit eigenvector of A corresponding to λ_j

Let also λ_ℓ be the eigenvalue of A closest to λ_j. Then, for large k,

$r(q^{(k)})$

$$\frac{\| q^{(k+1)} - v^{(j)} \|_2}{\| q^{(k)} - v^{(j)} \|_2} \approx \frac{|\lambda_j - \mu_k|}{|\lambda_\ell - \mu_k|}$$

$$\leq \frac{2 \|A\|_2 \, \| q^{(k)} - v^{(j)} \|_2}{|\lambda_\ell - \mu_k|}$$

goes to λ_j as $k \to \infty$

Hence,

$$\frac{\| q^{(k+1)} - v^{(j)} \|_2}{\| q^{(k)} - v^{(j)} \|_2^2} \lesssim \frac{2 \|A\|_2}{|\lambda_\ell - \lambda_j|}.$$