Further applications of SVD

Sensitivity of linear systems

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 10^{-8} \end{bmatrix} \qquad \begin{array}{l} \sigma_1 \approx \sqrt{2} \\ \sigma_2 \approx 10^{-8}/\sqrt{2} \end{array}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 10^{-8} \end{bmatrix} x = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 10^{-8} \end{bmatrix} \hat{x} = \underbrace{\begin{bmatrix} 1 \\ 10^{-8} \end{bmatrix}}_{\hat{b}} \qquad \hat{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\|\hat{b} - b\|_2 = 10^{-8} \ll \|\hat{x} - x\| = \sqrt{2}$$

The linear system $Ax = b$ is ill-conditioned.

In general, consider

$$Ax = b \quad \text{and} \quad A(x + \delta x) = (b + \delta b)$$

Absolute ~~cond~~ error

$$\|\delta x\|_2 / \|\delta b\|_2$$

$$A(x+\delta x) = (b+\delta b)$$

$$\Longrightarrow \cancel{Ax} + A(\delta x) = \cancel{b} + \delta b$$

$$\Longrightarrow \delta x = A^{-1}(\delta b)$$

$$\Longrightarrow (+) \|\delta x\|_2 = \|A^{-1}(\delta b)\|_2$$

$$\leqslant \|A^{-1}\|_2 \, \|\delta b\|_2$$

(Note: $(*) \|Aw\|_2 \leqslant \|A\|_2 \, \|w\|_2 \quad \forall w \in \mathbb{C}^n$

because $\|A\|_2 \geqslant \|A(w/\|w\|_2)\|_2 = \|Aw\|_2/\|w\|_2$

implying $(*)$)

Hence,

$$\|\delta x\|_2 \Big/ \|\delta b\|_2 \leqslant \sigma_1(A^{-1}) = \frac{1}{\sigma_n(A)}$$

Relative error

$$\frac{(\|\delta x\|_2 / \|x\|_2)}{(\|\delta b\|_2 / \|b\|_2)}$$

$$Ax = b$$

$$\Longrightarrow (++) \|A\|_2 \, \|x\|_2 \geqslant \|b\|_2$$

$$\Longrightarrow \quad \frac{1}{\|x\|_2} \leq \|A\|_2 \frac{1}{\|b\|_2}$$

Combining (+) and (++), we have

$$\frac{\|\delta x\|_2}{\|x\|_2} \leq \|A^{-1}\|_2 \|A\|_2 \frac{\|\delta b\|_2}{\|b\|_2}$$

$$\Longrightarrow \quad \frac{(\|\delta x\|_2 / \|x\|_2)}{(\|\delta b\|_2 / \|b\|_2)} \leq \|A^{-1}\|_2 \|A\|_2$$

Condition number of a matrix

$$K_2(A) = \|A^{-1}\|_2 \|A\|_2$$
$$= \sigma_1(A) / \sigma_n(A).$$

<u>Ex</u>

$$K_2\left(\begin{bmatrix} 1 & 1 \\ 0 & 10^{-8} \end{bmatrix}\right) \approx \frac{\sigma_1}{\sigma_2} = \frac{2}{10^{-8}}$$

A geometric view of a SVD

$A \in \mathbb{R}^{2\times 2}$ with SVD

$$A = [u_1 \;\; u_2] \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix}$$

Unit Ball

$$B := \{ w \in \mathbb{R}^2 \mid \|w\|_2 = 1 \}$$
$$= \{ \alpha_1 v_1 + \alpha_2 v_2 \mid \alpha_1^2 + \alpha_2^2 = 1 \}$$

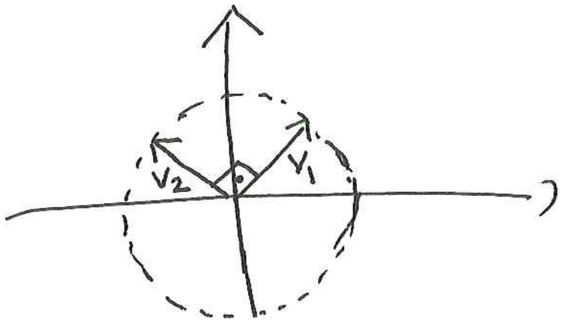


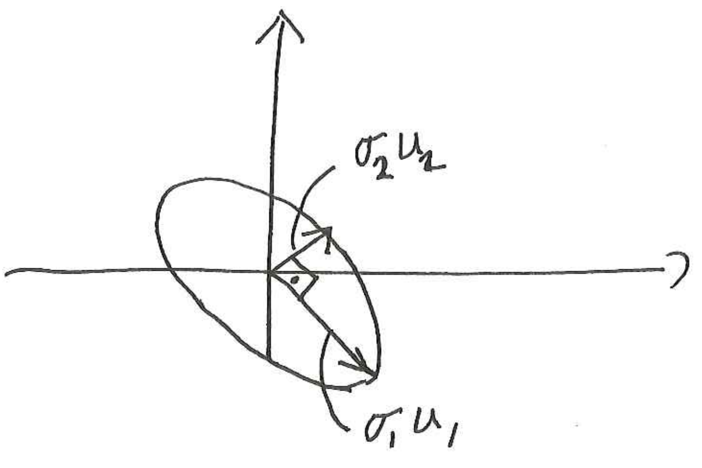
Image of the unit ball under $w \longmapsto Aw$

$$I := \{ Aw \in \mathbb{R}^2 \mid w \in B \}$$
$$= \{ Aw \mid w \in \mathbb{R}^2 \text{ s.t. } \|w\|_2 = 1 \}$$
$$= \{ A(\alpha_1 v_1 + \alpha_2 v_2) \mid \alpha_1, \alpha_2 \in \mathbb{R},\; \alpha_1^2 + \alpha_2^2 = 1 \}$$

$$I := \{ \underbrace{\alpha_1 \sigma_1}_{x :=} u_1 + \underbrace{\alpha_2 \sigma_2}_{y :=} u_2 \mid \alpha_1, \alpha_2 \in \mathbb{R}, \ \alpha_1^2 + \alpha_2^2 = 1 \}$$

$$= \{ x u_1 + y u_2 \mid x, y \in \mathbb{R}, \ \frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} = 1 \}$$

ellipsoid with principal semi-axes $\sigma_1 u_1$ and $\sigma_2 u_2$



In general, for $A \in \mathbb{C}^{m \times n}$ (assume $m \geq n$)

$$B := \{ w \in \mathbb{C}^n \mid \|w\|_2 = 1 \}$$

$$= \{ \alpha_1 v_1 + \cdots + \alpha_n v_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{C}, \ |\alpha_1|^2 + \cdots + |\alpha_n|^2 = 1 \}$$

$$I := \{ Aw \in \mathbb{C}^m \mid w \in B \}$$

$$= \{ A(\alpha_1 v_1 + \cdots + \alpha_n v_n) \mid \alpha_1, \ldots, \alpha_n \in \mathbb{C}, \ |\alpha_1|^2 + \cdots + |\alpha_n|^2 = 1 \}$$

$$= \{ \underbrace{\alpha_1 \sigma_1 u_1}_{z_1 :=} + \dots + \underbrace{\alpha_n \sigma_n u_n}_{z_n :=} \mid \alpha_1, \dots, \alpha_n \in \mathbb{C}, \ |\alpha_1|^2 + \dots + |\alpha_n|^2 = 1 \}$$

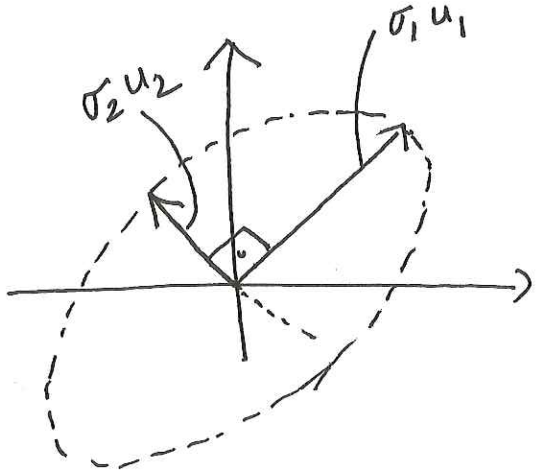
$$= \{ z_1 u_1 + \dots + z_n u_n \mid z_1, \dots, z_n \in \mathbb{C}, \ \frac{|z_1|^2}{\sigma_1^2} + \dots + \frac{|z_n|^2}{\sigma_n^2} = 1 \}$$

ellipsoid with principal semi-axes $\sigma_1 u_1, \dots, \sigma_n u_n$

Ex

$$A = \begin{bmatrix} 1 & -3 \\ 3 & -1 \end{bmatrix} \qquad \sigma_1 = 4 \quad \sigma_2 = 2$$

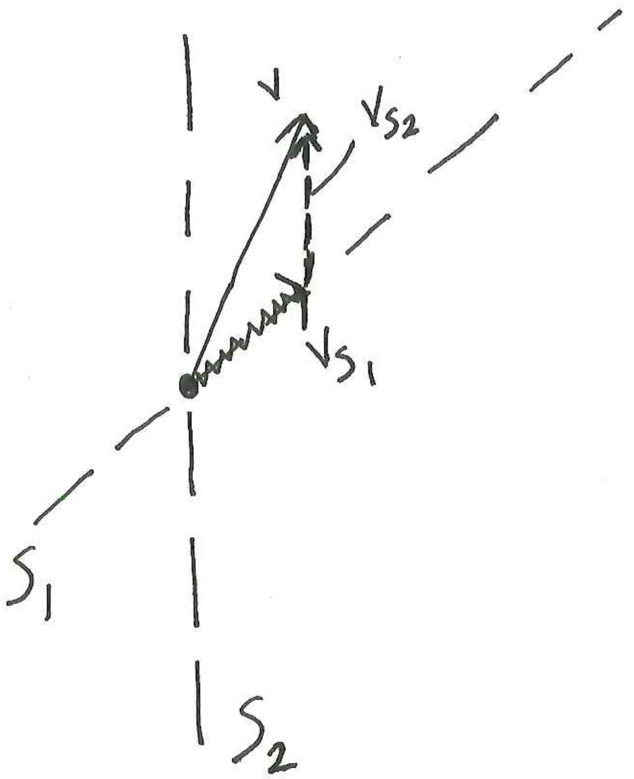
$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Projectors

S_1, S_2 subspaces of $\mathbb{C}^n$

$$S_1 \oplus S_2 = \mathbb{C}^n$$



$v = v_{S_1} + v_{S_2}$

$v_{S_1} \in S_1$

$v_{S_2} \in S_2$

v_{S_1} — projection of v onto S_1 along S_2

Ex

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ - projection of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ onto span$\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$ along span$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Proposition

Every $v \in \mathbb{C}^n$ can be written as

$$v = v_{S_1} + v_{S_2}$$

where $v_{S_1} \in S_1$, $v_{S_2} \in S_2$ in a unique way.

Proof

$$v = v_{S_1} + v_{S_2} = \tilde{v}_{S_1} + \tilde{v}_{S_2}$$

$$\exists\, v_{S_1}, \tilde{v}_{S_1} \in S_1 \text{ and } \exists\, v_{S_2}, \tilde{v}_{S_2} \in S_2$$

$$\Longrightarrow$$

$$v_{S_1} - \tilde{v}_{S_1} = \tilde{v}_{S_2} - v_{S_2} \in S_1 \cap S_2$$

$$\Longrightarrow \text{ (as } S_1 \cap S_2 = \{0\})$$

$$v_{S_1} = \tilde{v}_{S_1} \text{ and } v_{S_2} = \tilde{v}_{S_2} \qquad \square$$

Proposition

There exists $P \in \mathbb{C}^{n \times n}$ s.t.

$$Pv = v_{S_1} \qquad \forall v \in \mathbb{C}^n$$

The matrix P is called the projector onto S_1 along S_2.

Ex

Find the projector P onto $\operatorname{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$ along $\operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

It suffices to find the projections of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{P\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{P\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} P\left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}\right) &= \alpha_1 P\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_2 P\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \quad \forall \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \end{aligned}$$

$$\Longrightarrow \quad P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Observe $P^2 = P$.

<u>THM</u> (Characterization of Projectors)

(1) If $P \in \mathbb{C}^{n \times n}$ is a projector,
then $P^2 = P$.

(2) If $P \in \mathbb{C}^{n \times n}$ satisfies $P^2 = P$,
then P is a projector
onto $Col(P)$ along $Null(P)$.

<u>Proof</u>

(1) P is a projector

$$\Longrightarrow$$

For every v

$$Pv = v_{s_1} \quad \exists v_{s_1} \text{ s.t. } Pv_{s_1} = v_{s_1}$$

$$\Longrightarrow$$

For every v

$$Pv = v_{s_1} = Pv_{s_1} = P^2 v \quad \exists v_{s_1}$$

$$\Longrightarrow$$

$$P = P^2$$

(2) For every $v \in \mathbb{C}^n$

$$v = Pv + (I-P)v$$

It suffices to show

(i) $Col(P) \cap Col(I-P) = \{0\}$

(ii) $Col(I-P) = Null(P)$

(IT IS NOT ESSENTIAL TO READ THIS)

(i) Suppose $z \in Col(P) \cap Col(I-P)$.

$$z = P\underline{y} = (I-P)w \quad \exists\, \underline{y}, w.$$

$$\Longrightarrow$$

$$(+) \qquad Pz = (P-P^2)w = 0$$

and

$$(++) \qquad Pz = P^2 \cancel{z} = P\underline{y} = z$$

$\Longrightarrow$ (+) & (++) combined

$$z = 0.$$

(ii) Suppose $z \in Col(I-P)$.

$$z = (I-P)\underline{y} \quad \exists\, \underline{y}$$

$$\Longrightarrow Pz = (P-P^2)\underline{y} = 0$$

so $z \in Null(P)$.

Conversely suppose $z \in Null(P)$.

$$Pz = 0$$

$$\Longrightarrow z - Pz = \cancel{0}z$$

$$\Longrightarrow (I-P)z = z$$

so $z \in Col(I-P)$. $\square$

Ex

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Observe that $P^2 = P$

P is a projector

onto $\mathrm{Col}(P) = \mathrm{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$

along $\mathrm{Null}(P) = \mathrm{span}\left\{\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right\}$.