QR Factorization by Householder Reflectors

Every $A \in \mathbb{C}^{n \times p}$, $(n \geq p)$, has a (reduced) QR factorization of the form

$$A = \hat{Q}\hat{R}$$

$$\boxed{n\times p} = \boxed{n\times p}\ \boxed{p\times p}$$

$\hat{Q} \in \mathbb{C}^{n\times p}$ – has orthonormal columns

$\hat{R} \in \mathbb{C}^{p\times p}$ – upper triangular

Can be constructed by $\underbrace{\text{GS}}_{\text{Gram–Schmidt}}$ procedure (see Homework 1)

Full QR factorization

$$A = \underbrace{[\hat{Q}\ \tilde{Q}]}_{Q} \underbrace{\begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}}_{R}$$

$$\boxed{n\times p} = \boxed{n\times n}\ \boxed{n\times p}$$

$Q \in \mathbb{C}^{n \times n}$ – unitary

$R \in \mathbb{C}^{n \times p}$ – upper triangular

Ex

Reduced QR

$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{6} \end{bmatrix}$$

Full QR

$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{2} \\ 0 & -2/\sqrt{6} & 0 \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{6} \\ 0 & 0 \end{bmatrix}$$

Construction of (full) QR factorization by Householder reflectors, Overview

$$A = \begin{bmatrix} x & x & x & \cdots & x \\ x & x & x & & x \\ x & x & x & & x \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & & x \end{bmatrix} \xmapsto[\substack{\text{unitary} \\ Q_1 \text{ from left}}]{} \begin{bmatrix} x & x & x & \cdots & x \\ 0 & x & x & & x \\ 0 & x & x & & x \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & x & x & & x \end{bmatrix} = \underset{n \times n}{Q_1} A$$

$$\xmapsto[\substack{\text{unitary} \\ Q_2 \text{ from left}}]{} \begin{bmatrix} x & x & x & \cdots & x \\ 0 & x & x & & x \\ 0 & 0 & x & & x \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & x & & x \end{bmatrix} = \underset{n \times n}{Q_2} Q_1 A$$

$$\underbrace{Q_p \cdots Q_2 Q_1}_{Q\text{-unitary}} A = \underset{\substack{n \times p \text{ upper} \\ \text{triangular}}}{R}$$

Q_j – introduces 0s on the jth column below (j,j) entry.

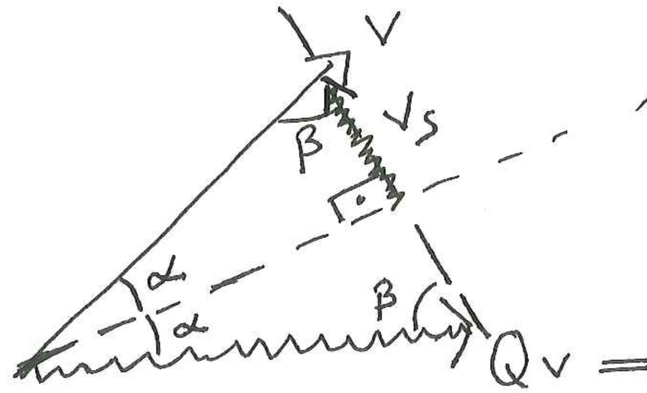
Householder Reflectors

$$v = \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix} \longmapsto \begin{bmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Qv = \|v\|_2 \, e_1$$

Householder reflector

1st column of $n \times n$ identity

$\mathrm{span}\{v - \|v\|_2 e_1\} = S$



$L = S^{\perp}$

$Qv = \|v\|_2 \, e_1$

$2\alpha + 2\beta = 180 \implies$ Right triangles

i.e. $L = S^{\perp}$

$$v = v_S + v_{S^\perp}$$

$$Qv = -v_S + v_{S^\perp}$$

$$= -v_S + (v - v_S) = v - 2v_S$$

that is

$$Qv = v - 2(qq^*)v$$

with $q = \{v - \|v\|_2 e_1\} \big/ \|v - \|v\|_2 e_1\|_2$

$$\boxed{\begin{aligned} Q &= I - 2qq^* \\ q &= \{v - \|v\|_2 e_1\} / \|v - \|v\|_2 e_1\|_2 \end{aligned}}$$

Ex

Find the QR factorization for

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$$

HH reflector that does

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \longmapsto \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$q = \left\{\begin{bmatrix} 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\} / \left\|\begin{bmatrix} 5 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \end{bmatrix}\right\|_2 = \frac{1}{\sqrt{20}}\begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{20}}\begin{bmatrix} 2 \\ -4 \end{bmatrix}\right)\left(\frac{1}{\sqrt{20}}\begin{bmatrix} 2 & -4 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{10}\begin{bmatrix} 4 & -8 \\ -8 & 16 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

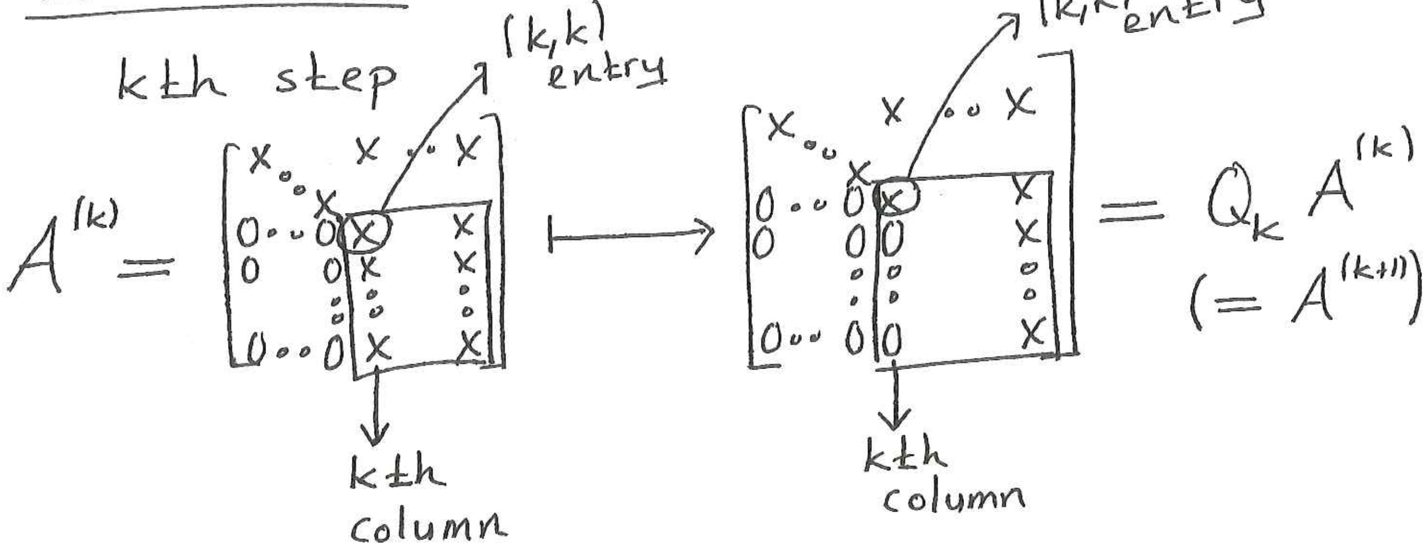
$$QA = \underbrace{\begin{bmatrix} 5 & 3.8 \\ 0 & 3.4 \end{bmatrix}}_{R} \Longrightarrow A = Q^* R = QR$$

$$= \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}\begin{bmatrix} 5 & 3.8 \\ 0 & 3.4 \end{bmatrix}$$

Remark $Q = I - 2qq^*$ is unitary (and Hermitian)

$$Q^*Q = (I - 2qq^*)(I - 2qq^*) = I - \cancel{4qq^*} + \cancel{4(qq^*)(qq^*)} = I$$

Algorithm



$$Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & \hat{Q}_k \end{bmatrix}$$

Householder reflector

$$\hat{Q}_k = I - 2 q_k q_k^*$$

where $$q_k = \frac{A^{(k)}(k:n,k) - \|A^{(k)}(k:n,k)\|_2 e_1}{\| A^{(k)}(k:n,k) - \|A^{(k)}(k:n,k)\|_2 e_1 \|_2}$$

Pseudocode

for $k = 1, \ldots, p$

$\quad h_k \leftarrow A(k:n,k)$

$\quad q_k \leftarrow \{h_k - \|h_k\|_2 e_1\} / \| h_k - \|h_k\|_2 e_1 \|_2$

$\quad A(k:n,k:p) \leftarrow A(k:n,k:p) - 2 q_k (q_k^* A(k:n,k:p))$

end

$R \leftarrow A$

At termination

$$Q_p \cdots Q_2 Q_1 A = R$$

$$\Longrightarrow A = (Q_1^* Q_2^* \cdots Q_p^*) R$$

$$= \underbrace{(Q_1 Q_2 \cdots Q_p)}_{Q\text{ - unitary}} R$$

$Q_1, Q_2, \cdots, Q_p, Q$ can be formed from $q_1, q_2, \cdots, q_p$.

\# of flops

$$\underbrace{q_k^* A(k:n, k:p)}_{1 \times (p-k+1)} \sim 2(n-k)(p-k) \text{ flops}$$

$$\underset{(n-k+1)\times 1}{(2q_k)} (q_k^* A(k:n, k:p)) \sim (n-k)(p-k) \text{ flops}$$

$$A(k:n, k:p) - (2q_k)(q_k^* A(k:n, k:p)) \sim (n-k)(p-k) \text{ flops}$$

$$\text{Total \# flops} \sim \sum_{k=1}^{p} 4(n-k)(p-k)$$

$$\sim \sum_{k=1}^{p-1} 4(n-p+k)k$$

$$= 4\frac{(n-p)(p-1)p}{2} + \frac{2}{3}(p-1)(p)(2p-1)$$

$$\sim 2np^2 - 2p^3/3$$