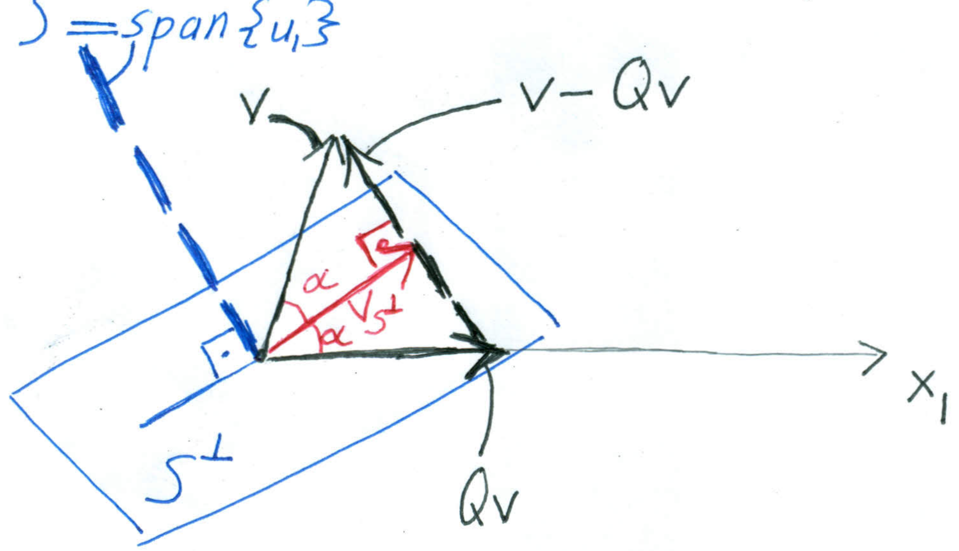
LECTURE 11

COMPUTATION OF THE QR FACTORIZATION BY HOUSEHOLDER REFLECTORS (CONTINUED)

Householder Reflectors in $\mathbb{C}^m$

Given $v \in \mathbb{C}^m$. Householder reflector $Q \in \mathbb{C}^{m\times m}$ associated with $v \in \mathbb{C}^m$ must satisfy

$$v \longrightarrow Qv = \|v\| e_1$$



* $u_1 = \dfrac{v - Qv}{\|v - Qv\|}$

* $S^\perp$ is the subspace orthogonal to u_1

Orthogonal decomposition of v

$$v = v_S + v_{S^\perp}$$

$Q \in \mathbb{C}^{m\times m}$ is the transformation that reflects about $S^\perp$

$$\begin{aligned} Qv &= -v_S + v_{S^\perp} \\ &= -v_S + (v - v_S) = v - 2v_S = v - 2(u_1 u_1^* v) \end{aligned}$$

$$= (I - 2u_1u_1^*)\,v$$

Householder reflector associated with $v \in \mathbb{C}^m$

$$Q = I - 2u_1u_1^*$$

where

$$u_1 = (v - Qv)/\|v - Qv\|$$

EXAMPLE

Find the Householder reflector associated with $v = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$.

① Determine u_1

$$u_1 = \frac{v - Qv}{\|v - Qv\|} = \frac{v - \overbrace{\|v\|e_1}^{\begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}}}{\|v - \|v\|e_1\|}$$

$$= \frac{1}{4}\begin{bmatrix} -2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

② Determine the reflector about $S^\perp$ (where $S = \text{span}\{u_1\}$)

$$Q = I - 2u_1u_1^*$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)\left(\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \end{bmatrix}\right)$$

$$\Longrightarrow \quad Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

③ Verify that $Qv = \|v\| e_1$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Algorithm to compute the QR factorization

Given

$$A = [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{C}^{m \times n} \quad \left(\begin{array}{c}\text{with} \\ m \geq n\end{array}\right)$$

STEP 1

$$A \longrightarrow Q_1 A = \left[\begin{array}{c|ccc} x & x & \cdots & x \\ \hline 0 & x & \cdots & x \\ \vdots & \vdots & & \vdots \\ 0 & x & \cdots & x \end{array}\right] \quad \underbrace{A^{(2)}}_{(m-1)\times(n-1)}$$

Q_1 is the Householder reflector associated with a_1

$$Q_1 = I - 2u_1u_1^* \quad \text{where} \quad u_1 = \frac{a_1 - \|a_1\|e_1}{\|a_1 - \|a_1\|e_1\|}$$

STEP 2

Repeat step 1 on $A^{(2)}$ (don't alter the first row and column of A)

$$A \longrightarrow Q_2 Q_1 A = \left[\begin{array}{cc|ccc} x & x & x & \cdots & x \\ 0 & x & x & \cdots & x \\ \hline 0 & 0 & x & \cdots & x \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & x & \cdots & x \end{array}\right] \quad \underbrace{A^{(3)}}_{(m-2)\times(n-2)}$$

Define

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_2 \end{bmatrix}$$

where $\hat{Q}_2$ is the Householder reflector associated with $a_1^{(2)}$ i.e.

$$\hat{Q}_2 = I - 2u_2u_2^* \qquad \text{and} \quad u_2 = \frac{a_1^{(2)} - \|a_1^{(2)}\|e_1}{\|a_1^{(2)} - \|a_1^{(2)}\|e_1\|}$$

STEP k $(k = 1, \dots, n-1)$

Let

$A^{(k)} \in \mathbb{C}^{(m-k+1)\times(n-k+1)}$ be lower right-most portion of $Q_{k-1}Q_{k-2}\dots Q_1A$.

$(k-1)\times(k-1)$: upper triangular

$$\underbrace{\begin{bmatrix} R & B \\ 0 & A^{(k)} \end{bmatrix}}_{Q_{k-1}\dots Q_1A} \longrightarrow \underbrace{\begin{bmatrix} R & B \\ 0 & \boxed{\begin{matrix} \times & \times & \cdots & \times \\ 0 & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ 0 & \times & \cdots & \times \end{matrix}} \end{bmatrix}}_{Q_kQ_{k-1}\dots Q_1A}$$

modified to introduce 0s on the kth column below diagonal

$\left(\begin{matrix} B \in \mathbb{C}^{(k-1)\times(n-k+1)} \text{ — full} \\ A^{(k)} \in \mathbb{C}^{(m-k+1)\times(n-k+1)} \text{ — full} \end{matrix}\right)$

Define

$$Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & \hat{Q}_k \end{bmatrix}$$

where $\hat{Q}_k \in \mathbb{C}^{(m-k+1)\times(m-k+1)}$ is the Householder reflector associated with $a_1^{(k)}$ i.e.

$$\hat{Q}_k = I - 2u_k u_k^* \quad \text{and} \quad u_k = \frac{a_1^{(k)} - \|a_1^{(k)}\| e_1}{\|a_1^{(k)} - \|a_1^{(k)}\| e_1\|}$$

<u>ALGORITHM</u> (QR factorization by Householder reflectors)

* Given $A \in \mathbb{C}^{m\times n}$ (with $m \geq n$)
* Produce a QR factorization $A = QR$ (Q: $m\times m$ unitary; R: $m\times n$ upper triangular)

for $k = 1, \dots, n-1$ (k: col #)

$v = A_{k:m,k}$

$u_k = (v - \|v\| e_1)/(\|v - \|v\| e_1\|)$

$A_{k:m,k:n} = A_{k:m,k:n} - 2u_k(u_k^* A_{k:m,k:n})$ (Modify $A^{(k)}$)

end

<u>REMARKS</u>

* At termination
  - (i) A contains the upper triangular factor R
  - (ii) $Q = Q_1 Q_2 \dots Q_n$ where

$$Q_k = \begin{bmatrix} I_{k-1} & 0 \\ 0 & I_{m-k+1} - 2u_k u_k^* \end{bmatrix}$$

* Order of operations to perform $2u_k u_k^* A_{k:m,k:n}$

EFFICIENT WAY: $(2u_k)(u_k^* A_{k:m,k:n})$ $\left(\begin{array}{c}\#\text{ FLOPS} \\ = \\ 3(m-k+1)\times(n-k+1) \\ +O(n)\end{array}\right)$

INEFFICIENT WAY: $2(u_k u_k^*)A_{k:m,k:n}$ $\left(\begin{array}{c}\#\text{ FLOPS} = \\ 2(m-k+1)^2\times(n-k+1) \\ +O(mn)+O(m^2)\end{array}\right)$

OPERATION COUNT

Operation	# FLOPS
$u_k = \dfrac{(v-\Vert v\Vert e_1)}{\Vert v-\Vert v\Vert e_1\Vert}$	$O(m)$
$A_{k:m,k:n} = A_{k:m,k:n} - 2u_k(u_k^* A_{k:m,k:n})$	$4(m-k+1)\times(n-k+1)+O(n)$

* TOTAL # FLOPS $= \displaystyle\sum_{k=1}^{n-1} 4(m-k+1)(n-k+1)+O(m)+O(n)$

$$= 4\left(mn(n-1) - m\frac{n(n-1)}{2} - n\frac{n(n-1)}{2} + \frac{n(n+1)(2n+1)}{6}\right)$$
$$+O(mn)+O(n^2)$$
$$= 2mn^2 - \frac{2n^3}{3} + O(mn) + O(n^2)$$

* If A is square $(m=n)$

TOTAL # FLOPS $= \dfrac{4n^3}{3} + O(n^2)$ (Gram-Schmidt would require $2n^3+O(n^2)$ flops)