LECTURE 13

PSEUDOINVERSE

Recall LSP

$$\underset{x \in \mathbb{C}^n}{\text{minimize}} \quad \|Ax - b\|_2 \qquad \left(\begin{array}{l} \text{where} \\ A \in \mathbb{C}^{m \times n} \\ b \in \mathbb{C}^m \\ m \geq n \end{array} \right.$$

Optimal $\hat{x} \in \mathbb{C}$ satisfies

$$\boxed{\text{NORMAL EQUATION}} \quad A^* A \hat{x} = A^* b$$

When A is full rank,

$$\hat{x} = \underbrace{(A^*A)^{-1}}_{n \times n} \underbrace{A^*}_{n \times m} b$$

We shall see that

$$A^+ = (A^*A)^{-1} A^* \in \mathbb{C}^{n \times m}$$

is a special case of the pseudoinverse. But pseudoinverse of A is defined even when A is not full rank.

ONE-SIDED INVERSES

Consider $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\Bigg(\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}}_{A^L - \text{Left Inverse}}\Bigg)\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

DEFN (Left Inverse)

Let $A \in \mathbb{C}^{m \times n}$ with $m \geq n$. Then $X \in \mathbb{C}^{n \times m}$ satisfying

$$XA = I_n$$

is called the left inverse of A.

DEFN (Right Inverse)

Let $A \in \mathbb{C}^{m \times n}$ with $m \leq n$. Then $X \in \mathbb{C}^{n \times m}$ satisfying

$$AX = I_m$$

is called the right inverse of A.

NOTATION

A^L – Left Inverse

A^R – Right Inverse

$$AX = I_m$$

$$\Longleftrightarrow$$

$$Ax_1 = e_1, \ \ldots, \ Ax_m = e_m$$

$$\left(\begin{array}{l} x_1, \ldots, x_m \text{ are cols of } X \\ e_1, \ldots, e_m \text{ are cols of } I_m \end{array} \right)$$

$$\Longleftrightarrow$$

$$\mathrm{Range}(A) = \mathbb{C}^m$$

$$\Longleftrightarrow$$

$$\mathrm{rank}(A) = m$$

THM (Existence of Right Inverse)
Let $A \in \mathbb{C}^{m \times n}$ with $m \leqslant n$. Then A^R exists if and only if $\mathrm{rank}(A) = m$.

THM (Existence of Left Inverse)
Let $A \in \mathbb{C}^{m \times n}$ with $m \geqslant n$. Then A^L exists if and only if $\mathrm{rank}(A) = n$.

Suppose A^R exists. Then it can be written of the form

$$(*)\ A^R = \underbrace{P}_{n\times n}\begin{bmatrix} \overset{B}{\cancel{A_1}} \\ \underset{C}{\cancel{A_2}} \end{bmatrix}\begin{array}{l} n\times n \\ (m-n)\times n \end{array}$$

where P (can be any invertible matrix, but here) is a permutation matrix.

<u>DEFN</u> (Permutation Matrix)

A matrix $P \in \mathbb{R}^{n\times n}$ is called a permutation matrix if along each column it has only one non-zero entry, and that non-zero entry is 1.

<u>EXAMPLE</u>

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ is a permutation matrix

Right multiplication permutes columns

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

Left multiplication permutes rows

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \end{bmatrix}$$

Since A^R exists, $\operatorname{rank}(A) = m$ that is A has m linearly independent columns.

Permute columns of A so that the first m columns are linearly independent, that is

$$(**) \quad AP = [\underset{\substack{m \times m \\ \operatorname{rank}(A_1)=m}}{A_1} \quad \underset{(n-m)\times m}{A_2}].$$

Choose P in $(*)$ as the permutation matrix above.

Now using $(*)$

$$
\begin{aligned}
I_m &= A A^R \\
&= AP \begin{bmatrix} B \\ C \end{bmatrix} \\
&= [A_1 \quad A_2] \begin{bmatrix} B \\ C \end{bmatrix} = A_1 B + A_2 C
\end{aligned}
$$

$$\Longrightarrow$$

$$B = A_1^{-1} - A_1^{-1} A_2 C$$

THM

Let $A \in \mathbb{C}^{m\times n}$ with $m \leqslant n$ (and $\operatorname{rank}(A)=m$). Then A^R is of the form

$$A^R = P\begin{bmatrix} A_1^{-1} - A_1^{-1}A_2 C \\ C \end{bmatrix}$$

where $P \in \mathbb{R}^{n\times n}$ is the permutation matrix defined by (**) and $C \in \mathbb{C}^{(n-m)\times m}$ is any matrix.

Similarly suppose $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$ and $\operatorname{rank}(A)=n$. Furthermore let $Q \in \mathbb{R}^{m\times m}$ be the permutation matrix such that

$$(***)\quad QA = \begin{bmatrix} A_3 \\ A_4 \end{bmatrix}, \quad A_3: n\times n,\ \operatorname{rank}(A_3)=n, \quad A_4: (m-n)\times n$$

that is Q permutes rows of A so that the first n rows are linearly independent.

THM

Let $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$ and $\operatorname{rank}(A)=n$. Then $A^L \in \mathbb{C}^{n\times m}$ is of the form

$$(+)\quad A^L = \begin{bmatrix} -DA_4A_3^{-1}+A_3^{-1} & D \end{bmatrix} Q$$

where Q is as defined by (***), and $D \in \mathbb{C}^{n\times(m-n)}$ is any matrix.

GENERALIZED INVERSE

One sided inverse does not exist, if A is not full rank

e.g. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ does not have any right inverse.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $\qquad$ are inconsistent.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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One sided inverse is not unique, when it exists

e.g. $$\left(\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}\right)\begin{bmatrix} \overbrace{\boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}}}^{A_3} \\ \underbrace{\boxed{\begin{matrix} 1 & 1 \end{matrix}}}_{A_4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\cancel{\begin{bmatrix} \frac{1}{2} & 5/2 \\ -1/2 & 5/2 \end{bmatrix}}$$

$$\begin{bmatrix} 1/2 & -1/2 & 1 \\ -1/2 & -1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \left(\begin{matrix} \text{Use } (+) \\ \text{with } D = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}\right)$$

Suppose A^R exists. Then

$$\underbrace{A A^R}_{m\times m} \underbrace{A}_{m\times n} = I_m \cdot A = A$$

$$\underbrace{A^R}_{n\times m} \underbrace{A A^R}_{m\times m} = A^R \cdot I_m = A^R$$

(Similarly for left inverse)

<u>DEFN</u> (Generalized Inverse)

Let $A \in \mathbb{C}^{m\times n}$. Then X satisfing

(i) $AXA = A$, and

(ii) $XAX = X$

is called the generalized inverse of A.

<u>NOTATION</u>

A^I – Generalized Inverse

Now suppose $A \in \mathbb{C}^{m\times n}$ with SVD of the form

$$A = \underbrace{U}_{m\times m} \underbrace{\Sigma}_{m\times n} \underbrace{V^*}_{n\times n}$$

$$(++) \qquad = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^*$$

($\Sigma_r \in \mathbb{R}^{r\times r}$ consisting of non-zero singular values)

$$= U_r \Sigma_r V_r^*$$

Let

$$X = V \begin{bmatrix} \Sigma_r^{-1} & \sqrt{\Sigma_r^{-1}}\, B_1 \\ B_2 \sqrt{\Sigma_r^{-1}} & B_2 B_1 \end{bmatrix} U^*$$

where $B_1 \in \mathbb{C}^{r \times (m-r)}$ and $B_2 \in \mathbb{C}^{(n-r) \times r}$ are any matrices.

Now

$$AX = U \begin{bmatrix} I_r & \sqrt{\Sigma_r}\, B_1 \\ 0 & 0 \end{bmatrix} U^*$$

and

$$\text{(i)}\ AXA = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^* = A$$

$$\text{(ii)}\ XAX = V \begin{bmatrix} \Sigma_r^{-1} & \sqrt{\Sigma_r^{-1}}\, B_1 \\ B_2 \sqrt{\Sigma_r^{-1}} & B_2 B_1 \end{bmatrix} U^* = X.$$

<u>THM</u> (Existence)

Let $A \in \mathbb{C}^{m \times n}$. There exists a generalized inverse of A. In particular any matrix of the form

$$(+{+}+)\ A^{I} = V \begin{bmatrix} \Sigma_r^{-1} & \sqrt{\Sigma_r^{-1}}\, B_1 \\ B_2 \sqrt{\Sigma_r^{-1}} & B_2 B_1 \end{bmatrix} U^*$$

for some $B_1 \in \mathbb{C}^{r \times (m-r)}$ and $B_2 \in \mathbb{C}^{(n-r) \times r}$ is a generalized inverse.

Now notice that

$$(AA^I)(AA^I) = (AA^IA)A^I = AA^I$$

$$(A^IA)(A^IA) = (A^IAA^I)A = A^IA,$$

that is

(i) AA^I and (ii) A^IA

are projectors.

LEMMA

Let A, B, C be matrices of appropriate sizes, and $C = AB$. Then

$$\operatorname{rank}(C) \leqslant \min(\operatorname{rank}(A), \operatorname{rank}(B))$$

PROOF

EXERCISE

LEMMA

For any generalized inverse A^I of A

$$\operatorname{rank}(A^I) = \operatorname{rank}(A).$$

PROOF

Since $A^I = A^IAA^I$, by previous lemma

$$\operatorname{rank}(A^I) \leqslant \min(\operatorname{rank}(A^I), \operatorname{rank}(AA^I))$$

$$\leqslant \min\left(\operatorname{rank}(A^I), \operatorname{rank}(A)\right)$$

$$\leqslant \min\left(\operatorname{rank}(A)\right).$$

Similarly since $A = AA^IA$,

$$\operatorname{rank}(A) \leqslant \min\left(\operatorname{rank}(A), \operatorname{rank}(A^IA)\right)$$

$$\leqslant \min\left(\operatorname{rank}(A), \operatorname{rank}(A^I)\right)$$

$$\leqslant \operatorname{rank}(A^I)$$

$\square$

<u>THM</u>

(1) AA^I is an (in general oblique) projector onto Range(A) along Null(A^I).

(2) A^IA is a projector onto Range(A^I) along Null(A).

<u>PROOF</u>

(1)

AA^I is a projector onto Range(AA^I) along Null(AA^I).

(i) Range(A) = Range(AA^I)

$$x \in \text{Range}(AA^I) \implies AA^I y = x \quad \exists y$$

$\boxed{\text{Range}(AA^I) \subseteq \text{Range}(A)}$

$$\implies Az = x \quad \exists z$$

$$\implies x \in \text{Range}(A)$$

$$x \in \text{Range}(A) \implies Az = x \quad \exists z$$

$\boxed{\text{Range}(A) \subseteq \text{Range}(AA^I)}$

$$\implies AA^I Az = x \quad \exists z$$

$$\implies AA^I y = x \quad \exists y$$

(ii) $\text{Null}(AA^I) = \text{Null}(A^I)$

$$x \in \text{Null}(A^I) \implies A^I x = 0$$

$\boxed{\text{Null}(A^I) \subseteq \text{Null}(AA^I)}$

$$\implies AA^I x = 0$$

$$\implies x \in \text{Null}(AA^I)$$

Furthermore

$$\text{rank}(\underbrace{AA^I}_{m\times m}) = \text{rank}(A) \quad \left(\begin{array}{c}\text{since} \\ \text{Range}(AA^I) = \text{Range}(A)\end{array}\right)$$

$$= \text{rank}(\underbrace{A^I}_{n\times m}) \quad \left(\begin{array}{c}\text{PREVIOUS} \\ \text{LEMMA}\end{array}\right)$$

$$\implies$$

$$\dim(\text{Null}(AA^I)) = m - \text{rank}(AA^I)$$

$$= m - \text{rank}(A^I)$$

$$= \dim(\text{Null}(A^I))$$

The facts

$$Null(A^I) \subseteq Null(AA^I)$$

and

$$\dim(Null(A^I)) = \dim(Null(AA^I))$$

imply

$$Null(A^I) = Null(AA^I)$$

$\blacksquare$

(2)

Proop is similar by establishing

(i) $\quad Range(A^I) = Range(A^I A)$

(ii) $\quad Null(A^I A) = Null(A)$ $\square$

EXAMPLE

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ does not have

* regular inverse A^{-1}, and
* one-sided inverses A^L and A^R.

It has the SVD

$$A = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)$$

For instance applying (+++) with $B_1 = 1/\sqrt{2}$ and $B_2 = 0$ yields

$$A^I = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1/2 & 1/\sqrt{2} \\ 0 & 0 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \cancel{2}\,1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

(Verify that $AA^IA = A$ and $A^IAA^I = A^I$)

Projectors

* $AA^I = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$ projects onto $\underbrace{\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}_{Range(A)}$ along $\underbrace{\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}_{Null(A^I)}$

* $A^IA = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ projects onto $\underbrace{\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}}_{Range(A^I)}$ along $\underbrace{\text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}}_{Null(A)}$