LECTURE 15

BACKWARD ERROR ANALYSIS (PART I)

(TREFETHEN & BAU 14-15)

Let

$\tilde{f}(x)$ denote the computed value of f with input $fl(x)$

We define

absolute error $\quad \| \tilde{f}(x) - f(x) \|$

relative error $\quad \dfrac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|}$

Typically a numerical algorithm is said to be accurate if for all x

$$\frac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|} = O(\epsilon_{mach})$$

that is there exists a constant c s.t.

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} \leq c\,\epsilon_{mach}$$

for all $\epsilon_{mach} > 0$ sufficiently close to zero.

In backward error analysis the relative error is determined using

CONCERNS THE PROBLEM * the relative condition number of f, and

CONCERNS THE ALGORITM * the backward error of the algorithm.

DEFN (Backward Error)
Suppose the algorithm satisfies

$$\tilde{f}(x) = f(x + \delta x)$$

for some δx. Then the quantity

$$\|\delta x\| / \|x\|$$

is called the backward error of the algorithm at x.

DEFN (Backward Stability)

An algorithm $\tilde{f}$ is called backward-stable if for all x there exists a δx such that

(i) $\tilde{f}(x) = f(x+\delta x)$

(ii) $\|\delta x\| / \|x\| = O(\epsilon_{mach})$

EXAMPLE

Later we will see that the matrix-vector product $f(A) = Ax \; (A \in \mathbb{R}^{n \times n})$ can be computed (in any floating point arithmetic) so that

$$\tilde{f}(A) = (A + \delta A)x$$

for some δA with

$$|(\delta A)_{ij}| \leq n \epsilon_{mach} a_{ij}$$

(i) Backward error in 1-norm

$$\|\delta A\|_1 \leq n \epsilon_{mach} \|A\|_1$$

(since $\|(\delta A)_j\|_1 \leq \sum_{i=1}^{n} n \epsilon_{mach} a_{ij} = n \epsilon_{mach} \|a_j\|_1$)

Backward error is at most $n\epsilon_{mach}$.

(ii) Backward stability

$$\frac{\|\delta A\|_1}{\|A\|_1} \leq n\epsilon_{mach} = O(\epsilon_{mach})$$

for all $A \in \mathbb{C}^{n\times n}$.

$f(A) = Ax$ can be computed in a numerically backward-stable manner.

REMARK

As in the example above in general

* the backward error typically increases as the size of the domain increases,

* in the expression

$$\|\delta x\|/\|x\| = O(\epsilon_{mach})$$

for backward stability the right-hand side can depend on the dimension of the domain.