LECTURE #15

CONDITION NUMBERS (PART I)

$$f : V \longrightarrow W$$

V, W are vector spaces.

Condition number measures the sensitivity of f w.r.t. perturbations in the input.

EXAMPLES

① Linear Systems

$$A \in \mathbb{C}^{n \times n} \quad \text{and} \quad b \in \mathbb{C}^n$$

$$Ax = b$$

Solution x as a function of A.

$$x : \mathbb{C}^{n \times n} \longrightarrow \mathbb{C}^n$$

$$x(A) = A^{-1} b \quad \left(\begin{matrix} \text{Assume} \\ A \text{ is invertible} \end{matrix} \right)$$

How does $x(A+\delta A)$ and $x(A)$ compare? (for small δA)

② Least Squares Problem

$A \in \mathbb{C}^{m\times n}$ (with $m>n$) and $b \in \mathbb{C}^m$.

Find x_* such that

$$\|Ax_* - b\|_2 = \underset{x\in\mathbb{C}^n}{\text{minimize}} \|Ax-b\|_2$$

Solution x_* as a function of b

$$x_* : \mathbb{C}^m \longrightarrow \mathbb{C}^n$$

$$x_*(b) = A^+ b$$

How does $x_*(b+\delta b)$ compare with $x(b)$? (for small δb)

③ Singular Value Problem

$A \in \mathbb{C}^{m\times n}$

Largest singular value as a function of A

$$\sigma_1 : \mathbb{C}^{m\times n} \longrightarrow \mathbb{R}$$

$$\sigma_1(A) = \underset{\substack{x\in\mathbb{C}^n \\ \text{s.t.} \\ \|x\|_2=1}}{\text{maximize}} \|Ax\|_2$$

Absolute Condition Number

$$\kappa = \lim_{\delta \to 0^+} \sup_{\|\delta x\| \leqslant \delta} \frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|}$$

REMARK

sup S : supremum (smallest upper bound) of S

e.g.

$\boxed{\text{ATTAINED AT } n=1}$ $\sup \left\{ \frac{2}{n^2} \;\middle|\; n \in \mathbb{Z}^+ \right\} = 2$

$\boxed{\text{IS NOT ATTAINED}}$ $\sup \left\{ \frac{x^2-1}{x^2} \;\middle|\; x \in \mathbb{R}^+ \right\} = 1$

EXAMPLES

① Polynomial Root Finding

Root of $p(x) = x^3 - \epsilon$ as a function of ϵ.

$r(\epsilon) : \mathbb{R} \to \mathbb{R}$

$r(\epsilon) = \sqrt[3]{\epsilon}$

Absolute Condition Number at $\epsilon = 0$

$$\frac{|r(\epsilon+\delta\epsilon) - r(\epsilon)|}{|\delta\epsilon|} = \frac{|\sqrt[3]{\delta\epsilon}|}{|\delta\epsilon|} \quad \left(\begin{array}{l}\text{where} \\ \epsilon = 0\end{array}\right.$$

$$= \frac{1}{\sqrt[3]{\delta\epsilon^2}}$$

$$\kappa = \lim_{\delta\to 0^+} \sup_{\substack{\delta\epsilon \\ \text{s.t.} \\ |\delta\epsilon| \leqslant \delta}} \frac{1}{\sqrt[3]{\delta\epsilon^2}}$$

$$= \lim_{\delta\to 0^+} \frac{1}{\sqrt[3]{\delta^2}} = \infty$$

(2) Matrix-Vector Product

$$f: \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^n$$

$$f(A) = Ax \quad \left(\begin{array}{l}\text{where} \\ x \in \mathbb{C}^n \text{ is fixed}\end{array}\right)$$

Absolute Condition Number at any $A \in \mathbb{C}^{n\times n}$.

$$\sup_{\|\delta A\| \leqslant \delta} \frac{\|f(A+\delta A) - f(A)\|}{\|\delta A\|}$$

$$= \sup_{\|\delta A\| \leq \delta} \frac{\|(A+\delta A)x - Ax\|}{\|\delta A\|}$$

$$= \sup_{\|\delta A\| \leq \delta} \frac{\|\delta A x\|}{\|\delta A\|} \quad \left(\begin{array}{l} \text{In general} \\ \|\delta A x\| \leq \|\delta A\| \, \|x\| \\ \text{Choose } \delta A \text{ s.t.} \\ \|\delta A x\| = \|\delta A\| \, \|x\| \end{array} \right.$$

$$= \sup_{\|\delta A\| \leq \delta} \frac{\|\delta A\| \, \|x\|}{\|\delta A\|} = \|x\|$$

Consequently

$$K = \lim_{\delta \to 0^+} \sup_{\|\delta A\| \leq \delta} \frac{\|f(A+\delta A) - f(A)\|}{\|\delta A\|}$$

$$= \|x\|$$

TERMINOLOGY

f is ill-conditioned

CONDITION NUMBER OF f IS LARGE

f is well-conditioned

CONDITION NUMBER OF f IS SMALL

Polynomial root-finding in general is ill-conditioned (e.g. Example 1)

Relative Condition Number

$$\tilde{\kappa} = \lim_{\delta \to 0^+} \sup_{\|\delta x\| \leqslant \delta} \frac{\|f(x+\delta x) - f(x)\| / \|f(x)\|}{\|\delta x\| / \|x\|}$$

$$= \kappa \frac{\|x\|}{\|f(x)\|}$$

EXAMPLE

Matrix-vector product

$$f : \mathbb{C}^{n \times n} \longrightarrow \mathbb{C}^n$$

$$f(A) = Ax$$

Relative condition number

$$\tilde{\kappa} = \kappa \frac{\|A\|}{\|Ax\|}$$

$$= \frac{\|x\| \, \|A\|}{\|Ax\|}$$

Assuming A is invertible

$$\|x\| = \|A^{-1} A x\| \leq \|A^{-1}\| \, \|Ax\| \implies \frac{\|x\|}{\|Ax\|} \leqslant \|A^{-1}\|$$

Consequently

$$\tilde{\kappa} \leqslant \|A^{-1}\| \, \|A\|$$

(For all A there exists an x s.t. $\|x\| = \|A^{-1}\| \, \|Ax\|$; then $\tilde{\kappa} = \|A^{-1}\| \, \|A\|$)

Condition Number of a Matrix

Let $A \in \mathbb{C}^{n \times n}$ be invertible.

$$\kappa(A) := \|A^{-1}\| \, \|A\|$$

REMARK

$\kappa(A)$ is large if either

* A has large entries, OR
* A is close to singularity, i.e., columns of A are almost linearly dependent.