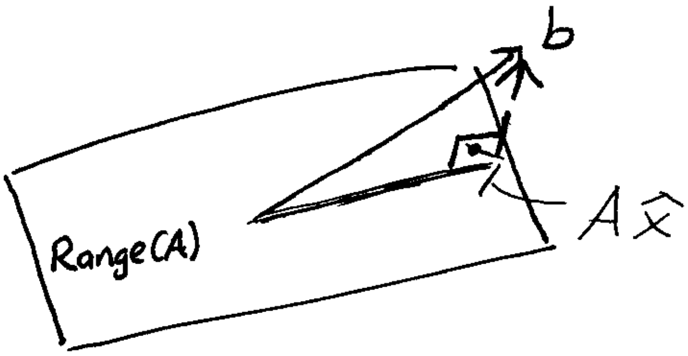
LECTURE 16

CONDITIONING OF THE LEAST SQUARES PROBLEM

Least Squares Problem

$$\| A\hat{x} - b \|_2 = \underset{x \in \mathbb{C}^n}{\text{minimize}} \; \|Ax - b\|_2$$

$A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$



Let $y := A\hat{x}$ (i.e. y is the orthogonal projection of b onto Range(A))

Questions

(1) How does $\hat{x}$ change when b is slightly changed?

(2) How does y change when b is slightly changed?

Sensitivity of $\underline{y}$ w.r.t. b

$$\underline{y} = AA^{+}b$$

View $\underline{y}$ as a function of b

$$\underline{y} : \mathbb{C}^m \longrightarrow \mathbb{C}^m \quad (A \text{ is fixed})$$

Absolute condition number

$$\kappa = \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leqslant \delta} \frac{\|\underline{y}(b+\delta b) - \underline{y}(b)\|}{\|\delta b\|}$$

$$= \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leqslant \delta} \frac{\|AA^{+}\delta b\|}{\|\delta b\|}$$

Choose δb in the direction of right svec assoc with ~~smallest~~ largest sval of AA^{+}

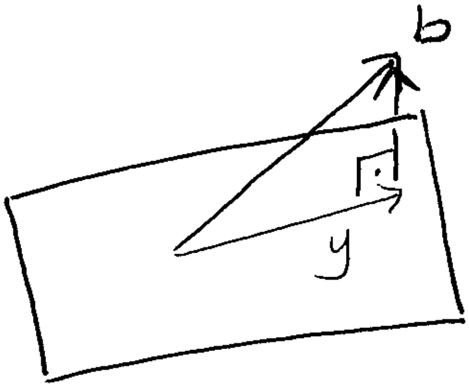
$$= \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leqslant \delta} \frac{\|AA^{+}\| \, \|\delta b\|}{\|\delta b\|}$$

$$= \|AA^{+}\| = 1$$

Relative condition number

$$\tilde{\kappa} = \frac{\kappa \, \cancel{\|\underline{y}(b)\|}^{\,\|b\|}}{\cancel{\|b\|}_{\,\|\underline{y}(b)\|}} = \frac{\|AA^{+}\|}{\cos\theta} = \frac{1}{\cos\theta}$$

where θ is the angle between $\underline{y}$ and b.



$$\theta := \arccos \frac{y}{b}$$

$$\left(\text{i.e. } \cos\theta = \frac{y}{b}\right)$$

REMARK

* y is very sensitive w.r.t. changes in b if y is nearly perpendicular to b.

Sensitivity of $\hat{x}$ w.r.t. b

Suppose $\hat{x}$ is the shortest solution to LSP, i.e.,

$$\hat{x} = A^+ b$$

Absolute condition number

$$K = \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leq \delta} \frac{\|\hat{x}(b+\delta b) - \hat{x}(b)\|}{\|\delta b\|}$$

$$= \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leq \delta} \frac{\|A^+ \delta b\|}{\|\delta b\|}$$

$$= \lim_{\delta \to 0^+} \sup_{\|\delta b\| \leq \delta} \frac{\|A^+\| \, \|\delta b\|}{\|\delta b\|} = \|A^+\|$$

Relative condition number

$$\tilde{\kappa} = \frac{\kappa \, \|b\|}{\|\hat{x}(b)\|}$$

$$= \frac{\|A^+\| \, \|A\|}{\|A\| \, \|\hat{x}(b)\|} \, \|b\|$$

$$= \kappa(A) \, \frac{\|y\|}{\|A\| \, \|\hat{x}(b)\|} \, \frac{\|b\|}{\|y\|}$$

$$= \kappa(A) \cdot \frac{\|A\hat{x}(b)\|}{\|A\| \, \|\hat{x}(b)\|} \cdot \frac{\|b\|}{\|y\|}$$

$$= \kappa(A) \cdot \frac{1}{\eta} \cdot \frac{1}{\cos\theta}$$

where

$$\eta := \frac{\|A\| \, \|\hat{x}(b)\|}{\|A\hat{x}(b)\|} \quad \left(\begin{array}{c}\text{Stretch factor} \\ \eta \geq 1 \\ \text{for induced norms}\end{array}\right)$$

$$\kappa(A) := \|A\| \, \|A^+\| \quad \left(\begin{array}{c}\text{condition number} \\ \text{of matrix } A\end{array}\right)$$

<u>REMARK</u>

* Shortest solution $\hat{x}$ to LSP is sensitive w.r.t. perturbations in b, when (i) matrix A is ill conditioned, OR (ii) y is nearly perpendicular to b

SUMMARY (Sensitivity of LSP w.r.t. b)

	κ	$\tilde{\kappa}$
y w.r.t. b	1	$1/\cos(\theta)$
$\hat{x}$ w.r.t. b	$\|A^+\|$	$\kappa(A)/\eta \cdot \cos(\theta)$

Sensitivity of LSP w.r.t. A

	$\tilde{\kappa}$
y w.r.t. A	$\kappa(A)/\cos(\theta)$
$\tilde{x}$ w.r.t. A	$\kappa(A)^2 \tan(\theta)/\eta + \kappa(A)$

(For derivations see Trefethen & Bau pages 133–135)