LECTURE 17

LINEAR SYSTEMS

A system of n linear equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

equivalently

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}
$$

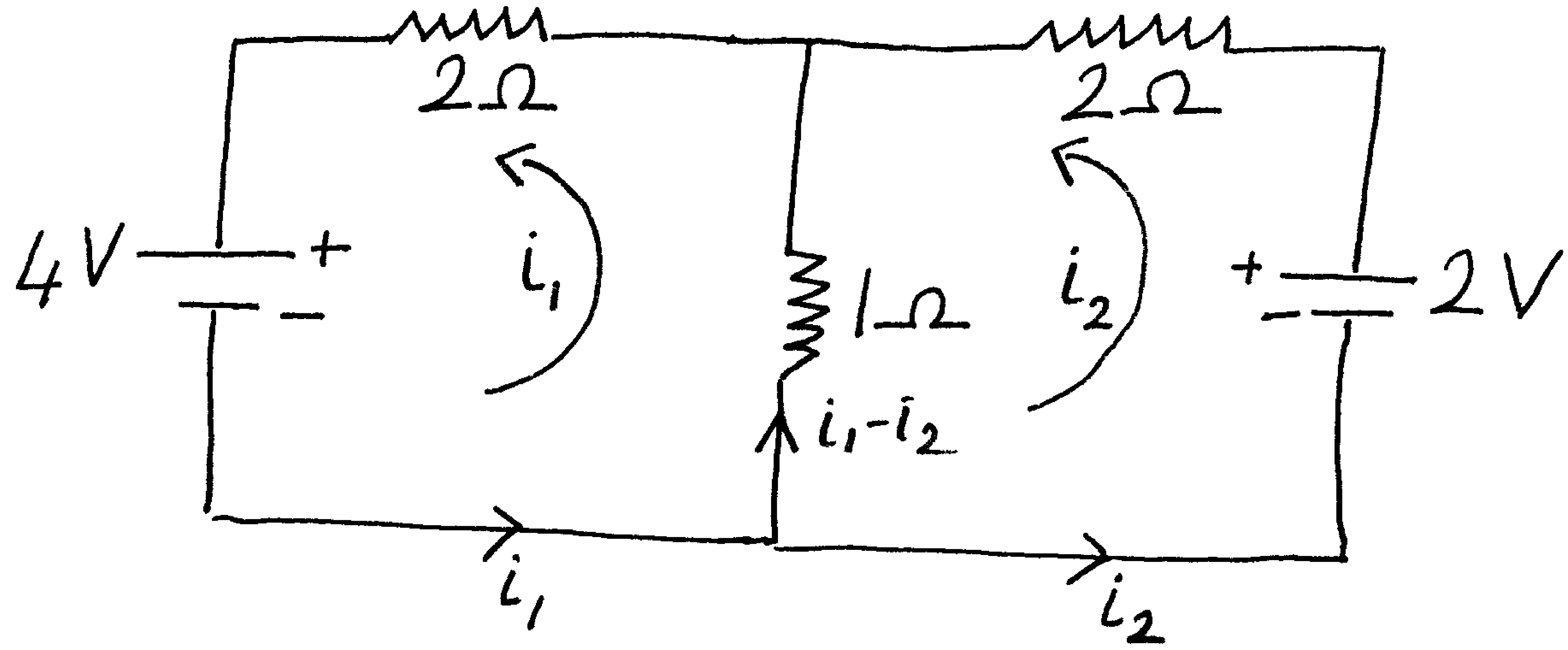
$A \in \mathbb{C}^{n \times n}$: coefficient matrix (given)

$b \in \mathbb{C}^{n}$: right-hand side vector (given)

$x \in \mathbb{C}^{n}$: vector of unknowns

EXAMPLES

① Electrical Circuits



Find the loop currents i_1 and i_2

CIRCUIT LAWS

(1) Around each mesh the total voltage drop is zero. (Kirchoff's Law)

(2) Around a resistor the voltage drop is

$$V = i \times r$$

i: current flowing through the resistor
r: resistance

(3) If the current flows from + to − across a voltage source, the voltage drop is +; otherwise −.

(4) The current through the resistors on the boundary of two meshes is the sum of currents for each mesh.

Left-hand mesh

$$1(i_1 - i_2) + 2i_1 + 4 = 0$$

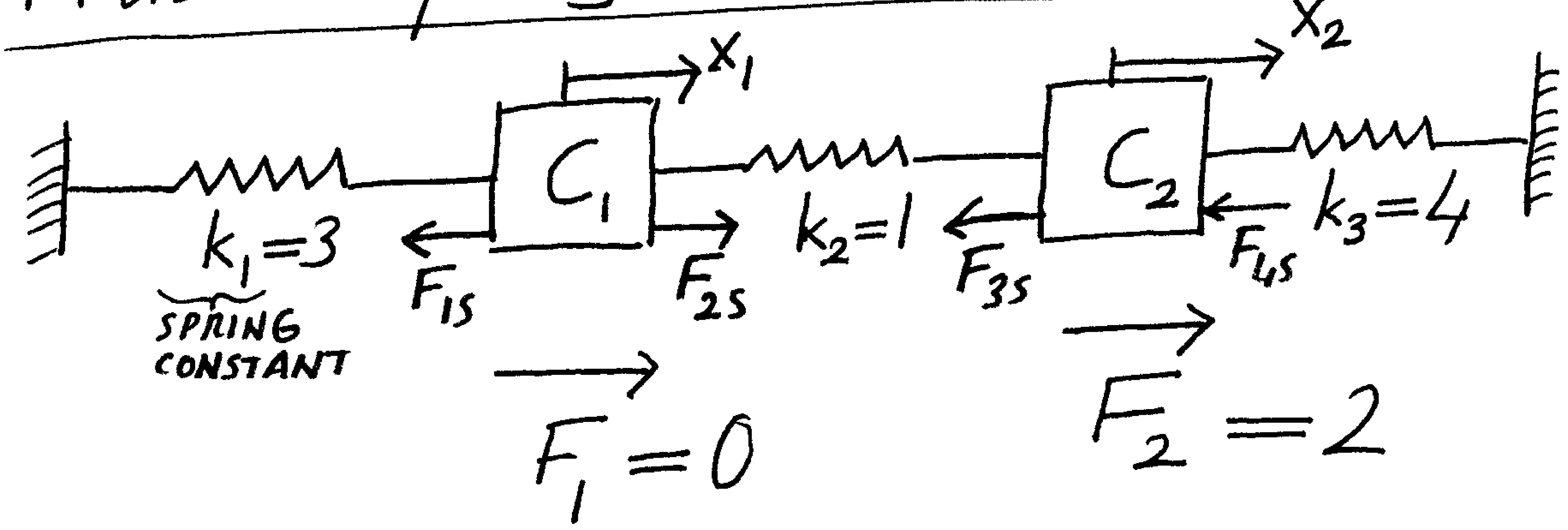
Right-hand mesh

$$-2 + 2i_2 + 1(i_2 - i_1) = 0$$

Resulting Linear System

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

2 Mass-Spring Systems



Find the equilibrium displacements x_1, x_2 (so that total forces on objects are zero.)

HOOKE'S LAW

Force applied by a spring

$$F_s = \underbrace{k}_{\text{spring constant}} \times \underbrace{x}_{\text{displacement of the spring}}$$

Forces on C_1

$$F_1 - F_{1s} + F_{2s} = 0$$
$$0 - 3x_1 + 1(x_2 - x_1) = 0$$

Forces on C_2

$$F_2 - F_{3s} - F_{4s} = 0$$
$$2 - 3(x_2 - x_1) - 4x_2 = 0$$

Resulting Linear System

$$\begin{bmatrix} -4 & 1 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

<u>ASSUMPTION</u>

Coefficient matrix A is invertible.

<u>THM</u> (Uniqueness of the solution)

$Ax = b$ has a unique soln

$$\Longleftrightarrow$$

A is invertible

<u>BACK SUBSTITUTION</u>

Procedure to solve upper triangular systems.

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$$

$$(1)\quad 3x_1 + x_2 + 2x_3 = 4$$
$$(2)\quad -2x_2 + x_3 = 8$$
$$(3)\quad 3x_3 = 6$$

From bottom to top

$$(3) \Longrightarrow x_3 = 2$$
$$(2) \Longrightarrow -2x_2 + 2 = 8 \Longrightarrow x_2 = -3$$
$$(1) \Longrightarrow 3x_1 - 3 + 4 = 4 \Longrightarrow x_1 = 1$$

In general $\left(A \text{ is invertible} \Longrightarrow a_{ii} \neq 0_{\,i=1,\dots,n}\right)$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ & 0 & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

(nth row) $\quad x_n = b_n / a_{nn}$

((n-1)st row) $\quad x_{n-1} = \left(b_{n-1} - a_{(n-1)n} x_n\right) / a_{(n-1)(n-1)}$

$\vdots$

(1st row) $\quad x_1 = \left(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n\right) / a_{11}$

General expression for x_i

$$x_i = \left(b_i - \sum_{j=i+1}^{n} a_{ij} x_j \right) / a_{ii}$$

<u>Back substitution</u> is this procedure of calculating unknowns starting from x_n to x_1.

<u>ALGORITHM</u> (Back Substitution)

Given upper triangular $A \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^n$

Compute x s.t. $Ax = b$

for $i = 1, \ldots, n$

    $x_i = b_i$

    for $j = i+1, \ldots, n$

        $\left. x_i = x_i - a_{ij} * x_j \right\}$ 2 Flops

    end

    $\left. x_i = x_i / a_{ii} \right\}$ 1 Flop

end

$$\text{TOTAL \# FLOPS} = \sum_{i=1}^{n} \left(\sum_{j=i+1}^{n} 2 \right) + 1$$

$$= \sum_{i=1}^{n} 2(n-i) + 1$$

$$= n + 2 \sum_{i=1}^{n-1} i$$

$$= n + \frac{\cancel{2}(n)(n-1)}{\cancel{2}} = n^2$$

FORWARD SUBSTITUTION

Procedure to solve a lower triangular system

$$Lx = b$$

where $L \in \mathbb{C}^{n \times n}$ is lower triangular.

Calculate the unknowns starting from x_1 to x_n (top to bottom).

TOTAL # Flops $= n^2$