LECTURE 18

GAUSSIAN ELIMINATION

$$\underset{\substack{n\times n \\ \text{(given)}}}{A}\ \underset{\substack{\text{in } \mathbb{C}^n \\ \text{(unknown)}}}{x} = \underset{\substack{\text{in } \mathbb{C}^n \\ \text{(given)}}}{b}$$

where A is nonsingular.

STRATEGY

(1) Convert $Ax=b$ into an equivalent upper triangular system $Ux=\hat{b}$

(2) Solve $Ux=\hat{b}$ by back substitution.

DEFN

The systems

$$Ax=b \quad \text{and} \quad \hat{A}x=\hat{b}$$

are called equivalent if they have exactly the same solution set.

Augmented Matrix: $[A \mid b]$ (for the system $Ax=b$)

Row Reduction

$$[A \mid b] \rightsquigarrow [U \mid \hat{b}]$$

upper triangular

apply row operations repeatedly

Row Operations

(1) Row Replace

Add a multiple of a row to another.

$$\left[\begin{array}{cc|c} 2 & 1 & -2 \\ -1 & 3 & 1 \end{array}\right] \xrightarrow[r_2 := r_2 + 2r_1]{} \left[\begin{array}{cc|c} 2 & 1 & -2 \\ 3 & 5 & -3 \end{array}\right]$$

(2) Row Scale

Multiply a row by a scalar.

$$\left[\begin{array}{cc|c} 2 & 1 & -2 \\ -1 & 3 & 1 \end{array}\right] \xrightarrow[\cancel{r_2 := r_2 + 3r_1}]{r_2 := -3r_2} \left[\begin{array}{cc|c} 2 & 1 & -2 \\ 3 & -9 & -3 \end{array}\right]$$

(3) Row Interchange

Swap two rows.

$$\left[\begin{array}{cc|c} 2 & 1 & -2 \\ -1 & 3 & 1 \end{array}\right] \xrightarrow[r_1 \leftrightarrow r_2]{} \left[\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & 1 & -2 \end{array}\right]$$

BASIC RESULT

Suppose

$$[A \mid b] \xrightarrow[\text{of the row operations}]{\text{apply one}} [\hat{A} \mid \hat{b}]$$

Then $Ax = b$ and $\hat{A}x = \hat{b}$ are equivalent.

$$[A \mid b] \xrightarrow[\text{operations repeatedly}]{\text{apply row}} [U \mid \hat{b}]$$

$$\Longrightarrow$$

$Ax = b$ and $Ux = \hat{b}$

are equivalent systems

ALGORITHM (Gaussian Elimination)

* Proceed column by column starting with first column moving to the right.

* Processing ith column

set every entry on the ith column below a_{ii} to zero by applying row replacement operations.

CASE $n=3$

$$\left[\begin{array}{ccc|c} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{array}\right] \underset{i=1}{\leadsto} \left[\begin{array}{ccc|c} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{array}\right]$$

$$\underset{i=2}{\leadsto} \left[\begin{array}{ccc|c} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{array}\right]$$

Solve by back substitution

EXAMPLE

Solve the following system by Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ -2 & 1 & -3 & 6 \\ 3 & 11 & 2 & 6 \end{array}\right] \xrightarrow[\substack{r_2 := r_2 + 2r_1 \\ r_3 := r_3 - 3r_1}]{} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 5 & -7 & 12 \\ 0 & 5 & 8 & -3 \end{array}\right]$$

$$\xrightarrow[r_3 := r_3 - r_2]{} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 5 & -7 & 12 \\ 0 & 0 & 15 & -15 \end{array}\right]$$

LU FACTORIZATION (WATKINS-1.7)

$$\underbrace{A}_{\substack{n\times n\\ \text{(nonsingular)}}} = \underbrace{L}_{\substack{n\times n\\ \text{(lower triangular)}}}\ \underbrace{U}_{\substack{n\times n\\ \text{(upper triangular)}}}$$

LU factorization can be computed by row-reducing A into U and storing multipliers

$$A=\begin{bmatrix} 2 & 3 & 1\\ -2 & 1 & 3\\ 6 & 1 & -1\end{bmatrix} \xrightarrow[\substack{r_2:=r_2+r_1\\ r_3:=r_3-3r_1}]{} \begin{bmatrix} 2 & 3 & 1\\ -1 & 4 & 4\\ 3 & -8 & -4\end{bmatrix}$$

$$\xrightarrow[r_3:=r_3+2r_2]{} \begin{bmatrix} 2 & 3 & 1\\ -1 & 4 & 4\\ 3 & -2 & 4\end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0\\ -1 & 1 & 0\\ 3 & -2 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 2 & 3 & 1\\ 0 & 4 & 4\\ 0 & 0 & 4\end{bmatrix}}_{U} = \begin{bmatrix} 2 & 3 & 1\\ -2 & 1 & 3\\ 6 & 1 & -1\end{bmatrix}$$

(unit lower triangular, i.e. entries on the diagonal are one)

FORMAL DESCRIPTION

NOTATION:

$A^{(k)}$: Matrix obtained from A after processing columns $i=1,\dots,k-1$

$\bar{a}_j^{(k)}$: jth row of $A^{(k)}$

$\bar{a}_j$: jth row of A

$\bar{u}_j$: jth row of U

$$\underbrace{\begin{bmatrix} 2 & 3 & +1 \\ -2 & 1 & 3 \\ 6 & 1 & -1 \end{bmatrix}}_{A=A^{(1)}} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & -8 & -4 \end{bmatrix}}_{A^{(2)}} \rightsquigarrow \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}}_{A^{(3)}=U}$$

$\bar{a}_3^{(2)} = [0 \ \ -8 \ \ -4]$ $\quad$ $\bar{a}_1^{(1)} = [2 \ \ 3 \ \ 1]$

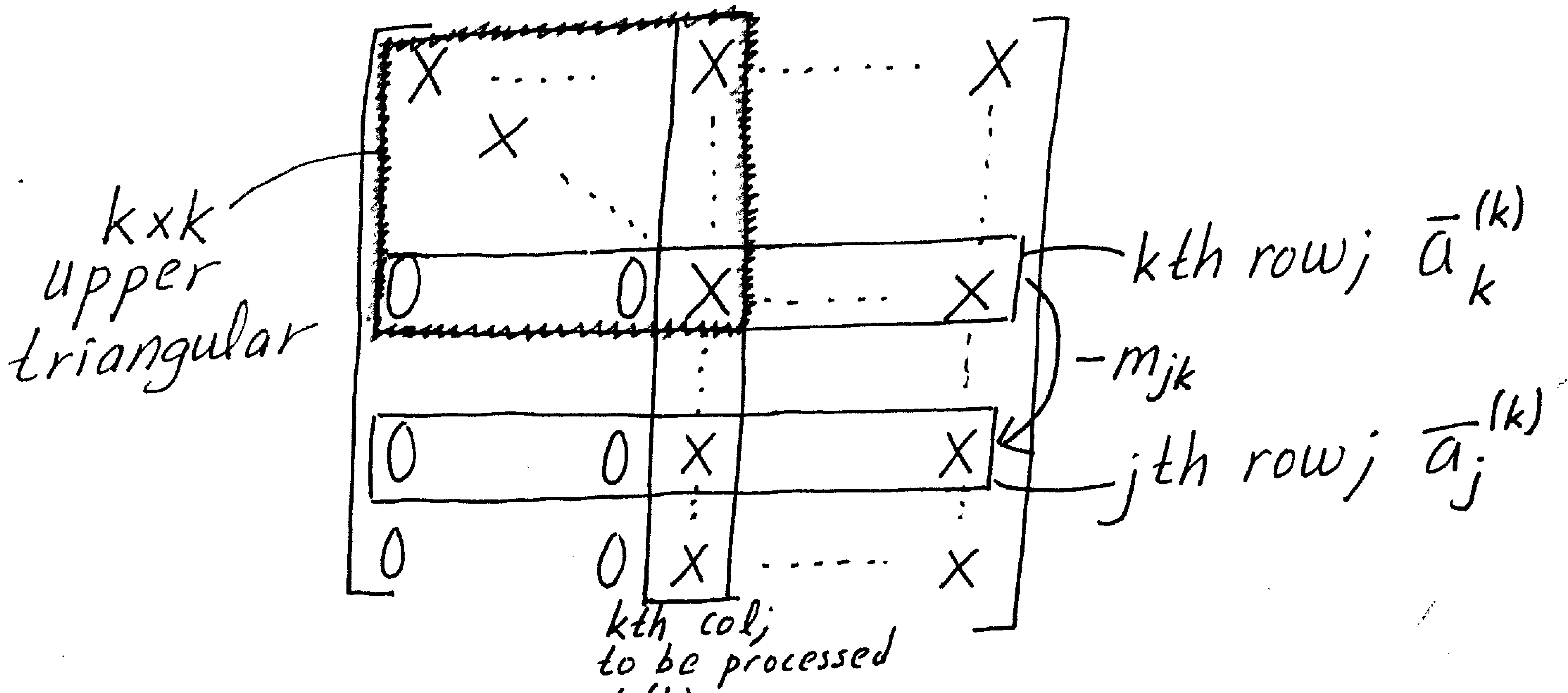
Note that

$$\bar{a}_1^{(1)} = \bar{u}_1 = [2 \ \ 3 \ \ 1]$$
$$\bar{a}_2^{(2)} = \bar{u}_2 = [0 \ \ 4 \ \ 4]$$
$$\bar{a}_3^{(3)} = \bar{u}_3 = [0 \ \ 0 \ \ 4]$$

BASIC PROPERTIES

(i) $\bar{u}_j = a_j^{(j)} \qquad j=1,\dots,n$

(ii) $\bar{a}_j = a_j^{(1)} \qquad j=1,\dots,n$ (since $A = A^{(1)}$)



$A^{(k)}$

Define the multiplier m_{jk} (for $j>k$) such that

$$(*) \quad \bar{a}_j^{(k+1)} = \bar{a}_j^{(k)} - m_{jk}\,\bar{a}_k^{(k)}$$

ALGORITHM (LU Factorization)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \xrightarrow[\text{row reduction}]{} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ m_{21} & u_{22} & \cdots & u_{2n} \\ m_{31} & m_{32} & \ddots & \\ m_{n1} & m_{n2} & m_{n(n-1)} & u_{nn} \end{bmatrix}$$

ASSUMPTION: Row reduction can be performed only by applying row replace operation repeatedly

PROPOSITION

$$A = \underbrace{\begin{bmatrix} 1 & & & 0 \\ m_{21} & 1 & & \\ m_{31} & m_{32} & \ddots & \\ \vdots & & & \\ m_{n1} & m_{n2} & \cdots & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & & u_{2n} \\ & 0 & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix}}_{U}$$

L (unit lower triangular; lower triangular with 1s on the diagonal)

U (upper triangular)

LEMMA

For all k, j such that $k \leqslant j$

$$\bar{a}_j^{(k)} = \bar{a}_j - \sum_{i=1}^{k-1} m_{ji} \bar{u}_i .$$

PROOF

The proof is by induction on k. (j is fixed)

Base Case ($k=1$)

$$\bar{a}_j^{(1)} = \bar{a}_j \quad \left(\text{By property (ii), since } A = A^{(1)}\right)$$

Inductive Case ($k \geqslant 2$)

Assume $\bar{a}_j^{(k-1)} = \bar{a}_j - \sum_{i=1}^{k-2} m_{ji} \bar{u}_i$ (Inductive hypothesis)

By equation (*) on page 7

$$a_j^{(k)} = a_j^{(k-1)} - m_{j(k-1)} \bar{a}_{k-1}^{(k-1)}$$

From the inductive hypothesis and property (i)

$$a_j^{(k)} = \underbrace{\left(\bar{a}_j - \sum_{i=1}^{k-2} m_{ji} \bar{u}_i \right)}_{a_j^{(k-1)}} - \underbrace{m_{j(k-1)} \bar{u}_{k-1}}_{m_{j(k-1)} \bar{a}_{k-1}^{(k-1)}}$$

$$= \bar{a}_j - \sum_{i=1}^{k-1} m_{ji} \bar{u}_i$$

$\square$

PROOF OF THE PROPOSITION

Apply lemma for $k=j$ to obtain

$$\underbrace{\bar{a}_j^{(j)}}_{\bar{u}_j} = \bar{a}_j - \sum_{i=1}^{j-1} m_{ji} \bar{u}_i$$

$$\Longrightarrow \bar{a}_j = \sum_{i=1}^{j-1} m_{ji} \bar{u}_i + \bar{u}_j$$

$$= \begin{bmatrix} m_{j1} & m_{j2} & \cdots & m_{j(j-1)} & 1 \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_j \end{bmatrix}$$

$$= \begin{bmatrix} m_{j1} & m_{j2} & \cdots & m_{j(j-1)} & 1 & 0 & \cdots & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{bmatrix}}_{U}$$

Combine the rows of A

$$A = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & & & 0 \\ m_{21} & 1 & & \\ \vdots & & \ddots & \\ m_{n1} & m_{n2} & \cdots\; m_{n(n-1)} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{bmatrix}}_{U}$$

$\square$

PSEUDOCODE TO COMPUTE LU FACTORIZATION

NOTATION

$a_{j,m_1:m_2}$: jth row of A restricted to the entries with column numbers in $[m_1, m_2]$

e.g. $A = \begin{bmatrix} 5 & -2 & 7 & 8 \\ 1 & -3 & 2 & 4 \end{bmatrix}$ $\quad a_{2,2:3} = [-3 \;\; 2]$

$a_{2,2:4} = [-3 \;\; 2 \;\; 4]$

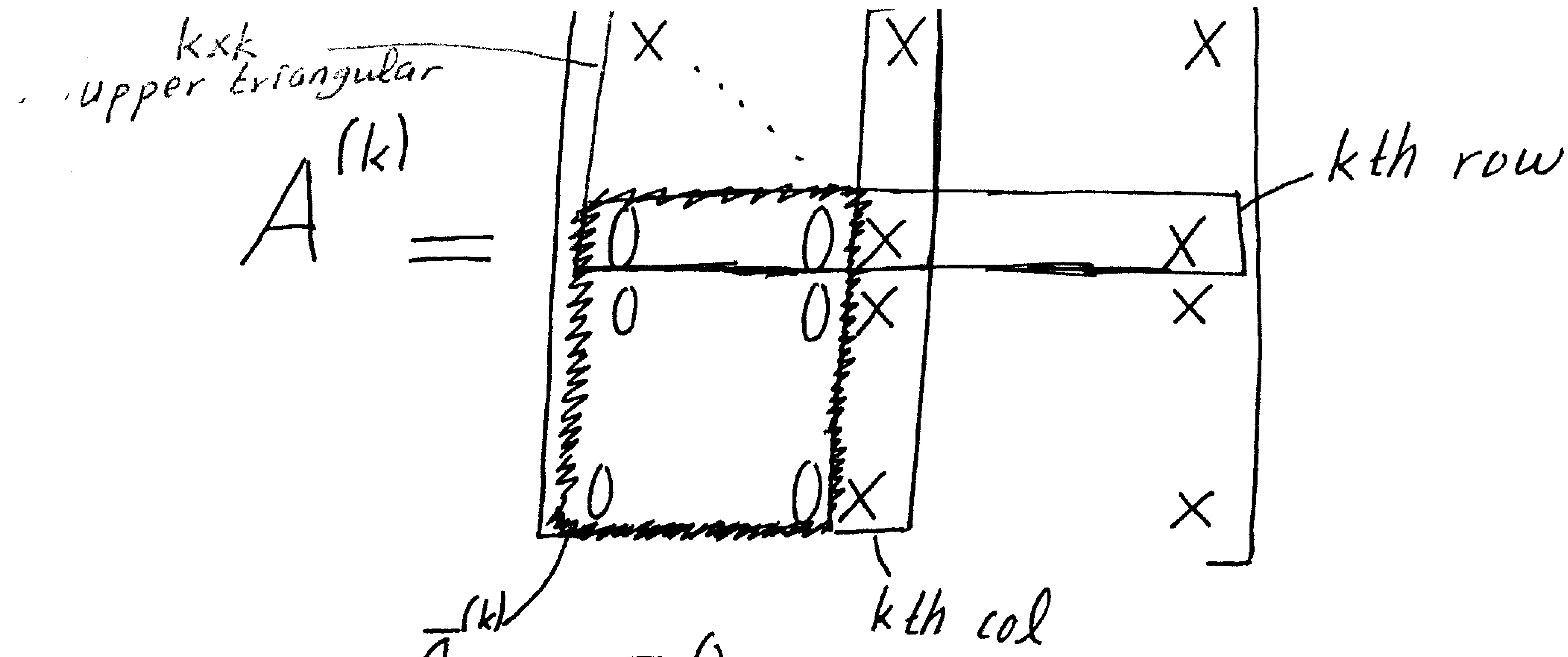
To process kth column we need to perform

$$(**) \quad a_j^{(k+1)} = a_j^{(k)} - m_{jk}\, a_k^{(k)} \quad \text{for } j = k+1, \dots, n$$

But note

(1) $a_{j,1:k-1}^{(k)} = 0$ for $j = k, k+1, \dots, n$

(2) we choose m_{jk} so that $a_{j,k}^{(k+1)} = 0$ for $j = k+1, \dots, n$



$\bar{a}^{(k)}_{j,1:k-1} = 0$
for $j = k, \dots, n$

Instead of $(**)$ perform

$$a^{(k+1)}_{j,k+1:n} = a^{(k)}_{j,k+1:n} - m_{jk}\, a^{(k)}_{k,k+1:n}$$

<u>PSEUDOCODE</u> (LU Factorization)

for $k = 1, \dots, n-1$ (k: col to be processed)
  for $j = k+1, \dots, n$ (j: row number)
    $m_{jk} \longleftarrow a_{jk} / a_{kk}$ } 1 flop
    $a_{j,k+1:n} \longleftarrow a_{j,k+1:n} - m_{jk}\, a_{k,k+1:n}$ } $2(n-k)$ flops
    $a_{jk} \longleftarrow m_{jk}$ (Store multiplier below the main diagonal)
  end
end

OPERATION COUNT

For $k=1,\dots,n-1$ and $j=k+1,\dots,n$

(1) a division is required for m_{jk}

(2) $2(n-k)$ flops are required to perform

$$a_{j,k+1:n} \leftarrow a_{j,k+1:n} - m_{jk}\, a_{k,k+1:n}$$

$$\text{TOTAL FLOPS} = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \left(2(n-k)+1\right)$$

$$= \sum_{k=1}^{n-1} \left(2(n-k)^2 + (n-k)\right)$$

$$= 2\left((n-1)^2 + (n-2)^2 + \dots + 1^2\right)$$
$$+$$
$$\left((n-1) + (n-2) + \dots + 1\right)$$

$$= 2\,\frac{(n-1)\,n\,(2n-1)}{6} + \frac{n(n-1)}{2}$$

$$= \frac{2n^3}{3} + O(n^2)$$

LU FACTORIZATION TO SOLVE LINEAR SYSTEMS

Consider

$$\begin{bmatrix} 2 & 3 & 1 \\ -2 & 1 & 3 \\ 6 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 11 \end{bmatrix}$$

equivalently

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\hat{x}} = \begin{bmatrix} 9 \\ -5 \\ 11 \end{bmatrix}$$

(1) FORWARD STAGE

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 11 \end{bmatrix}$$

Solve by forward substitution

$\hat{x}_1 = 9, \quad \hat{x}_2 = 4, \quad \hat{x}_3 = -8$

(2) BACKWARD STAGE

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

Solve by back substitution

$x_3 = -2, \quad x_2 = 3, \quad x_1 = 1$

PROCEDURE (To Solve $Ax=b$)

\# FLOPS

(1) Compute the LU-factorization $\frac{2n^3}{3} + O(n^2)$

$$A = LU$$

(2) Define $\hat{x} := Ux$. Solve n^2

$$L\hat{x} = b$$

by forward substitution

(3) Solve n^2

$$Ux = \hat{x}$$

by back substitution