MATH 504
FALL 2011

LECTURE 18

LINEAR SYSTEMS

n linear equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n
\end{aligned}
$$

Equivalently

$$
\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x}
=
\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}
$$

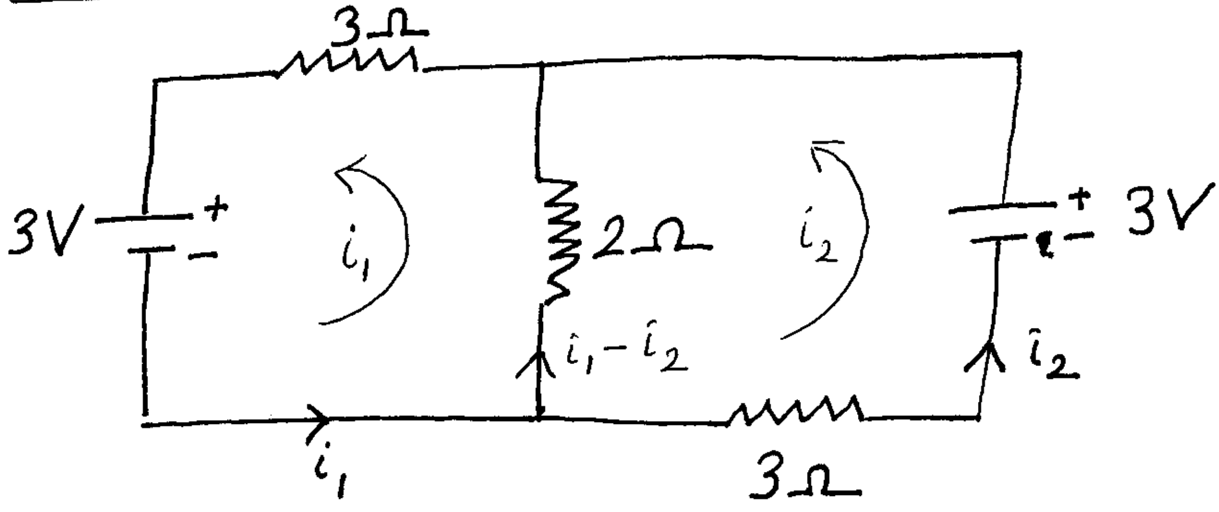
$A \in \mathbb{C}^{n\times n}$: coefficient matrix

$b \in \mathbb{C}^n$: right-hand vector

$x \in \mathbb{C}^n$: vector of unknowns

EXAMPLE

Electrical Circuits



Loop currents i_1 and i_2

CIRCUIT LAWS

(1) Around each mesh total voltage drop is zero. (Kirchoff's Law)

(2) Around a resistor the voltage drop

$$V = \underbrace{i}_{\text{CURRENT THR RESISTOR}} \cdot \underbrace{r}_{\text{RESISTANCE}}$$

Left-hand mesh

$$2(i_1 - i_2) + 3i_1 + 3 = 0$$

Right-hand mesh

$$3i_2 - 3 - 2(i_1 - i_2) = 0$$

Resulting Linear System

$$\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

General Assumption

$A \in \mathbb{C}^{n \times n}$ is invertible

$$\Longleftrightarrow$$

$Ax = b$ has a unique soln.

General Strategy (Numerical Solution)

(1) Convert $Ax = b$ into an

EXPENSIVE equivalent upper triangular system

$$\underbrace{U}_{\text{UPPER TRIANGULAR}} x = \hat{b}$$

CHEAP (2) Solve $Ux = \hat{b}$ by back substitution.

REMARK

$Ax = b$ and $Ux = \hat{b}$

are equivalent means
they have the same solution set.

BACK SUBSTITUTION

Procedure to solve upper triangular systems (A is invertible $\Longleftrightarrow a_{jj} \neq 0$)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & a_{2n} \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\boxed{n\text{th row}}$ $\quad x_n = b_n / a_{nn}$

$\boxed{(n-1)\text{st row}}$ $\quad x_{n-1} = (b_{n-1} - a_{(n-1)n} x_n) / a_{(n-1)(n-1)}$

$\boxed{(n-2)\text{nd row}}$ $\quad x_{n-2} = \dfrac{(b_{n-2} - a_{(n-2)n} x_n - a_{(n-2)(n-1)} x_{n-1})}{a_{(n-2)(n-2)}}$

COMPUTATION ORDER $\quad \vdots$

$\boxed{1\text{st row}}$ $\quad x_1 = \left(b_1 - \sum_{k=2}^{n} a_{1k} x_k\right) / a_{11}$

General expression for $x_j \; (j = 1, \ldots, n)$

$$x_j = \left(b_j - \sum_{k=j+1}^{n} a_{jk} x_k\right) / a_{jj}$$

Algorithm (Back Substitution)

Input : Upper triangular $A \in \mathbb{C}^{n \times n}$ and $b \in \mathbb{C}^n$

Output : x s.t. $Ax = b$

for $j = n$ down to 1

$\quad x_j = b_j$

$\quad$ for $k = j+1, \dots, n$

$\quad\quad x_j = x_j - x_k \cdot a_{jk} \}$ 2 FLOPS

$\quad$ end

$\quad x_j = x_j / a_{jj} \}$ 1 FLOP

end

$$\text{TOTAL \# FLOPS} = \sum_{j=1}^{n} \left(\sum_{k=j+1}^{n} 2 \right) + 1$$

$$= \sum_{j=1}^{n} 2(n-j) + 1$$

$$= 2\left(\sum_{j=0}^{n-1} j \right) + n$$

$$= 2\left(\frac{(n-1) \cdot n}{2} \right) + n$$

$$= n^2$$

FORWARD SUBSTITUTION

Procedure to solve lower triangular systems

$$\begin{bmatrix} a_{11} & & & 0 \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$\boxed{\text{1st row}}$ $\quad x_1 = b_1 / a_{11}$

$\boxed{\text{2nd row}}$ $\quad x_2 = (b_2 - a_{21} x_1) / a_{22}$

$\boxed{\text{3rd row}}$ $\quad x_3 = (b_3 - a_{31} x_1 - a_{32} x_2) / a_{33}$

$\vdots$

$\boxed{\text{nth row}}$ $\quad x_n = \left(b_n - \sum_{k=1}^{n-1} a_{nk} x_k\right) / a_{nn}$

General expression for x_j

$$x_j = \left(b_j - \sum_{k=1}^{j-1} a_{jk} x_k\right) / a_{jj}$$

Algorithm is similar to back-substitution with total # flops $= n^2$.

GAUSSIAN ELIMINATION

Procedure to reduce A into upper triangular form by applying row operations to the augmented matrix $[A \mid b]$.

Row Operations

① Row scale
Multiply a row by a scalar.

② Row interchange
Swap two rows

③ Row replace
Add a multiple of a row to another

REMARK

Row operations preserve the solution

EXAMPLE

$$\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ -2 & 1 & -3 & 6 \\ 3 & 11 & 2 & 6 \end{array}\right]}_{\text{Augmented matrix}} \xrightarrow[r_3 := r_3 - 3r_1]{r_2 := r_2 + 2r_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 5 & -7 & 12 \\ 0 & 5 & 8 & -3 \end{array}\right]$$

$$\xrightarrow[r_3 := r_3 - r_2]{} \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & -2 & 3 \\ 0 & 5 & -7 & 12 \\ 0 & 0 & 15 & -15 \end{array}\right]}_{\text{Solve by back substitution}}$$

<u>Outline</u> (Gaussian elimination)

* Proceed column by column from left to right.
* Processing ith column
  + Set entries below a_{ii} on the ith column by applying row-replace operations.

A TRIANGULAR TRIANGULAZITION POINT OF VIEW — LU FACTORIZATION

Recall that QR factorization by HH reflectors is an orthogonal triangularization.

$$A \longrightarrow Q_1 A = \begin{bmatrix} \times & \times & \cdots & \times \\ 0 & \times & & \times \\ \vdots & \vdots & & \\ 0 & \times & & \times \end{bmatrix}$$

$\begin{pmatrix} Q_1, \dots, Q_{n-1} \\ \text{are unitary} \end{pmatrix}$

$$\longrightarrow Q_2 Q_1 A = \begin{bmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & & \times \\ 0 & 0 & \times & & \times \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & \times & & \times \end{bmatrix}$$

$$\longrightarrow Q_{n-1} \cdots Q_2 Q_1 A = \underbrace{\begin{bmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & & \\ 0 & 0 & \times & & \\ \vdots & \vdots & 0 & \ddots & \\ 0 & 0 & & & \times \end{bmatrix}}_{R \ (\text{upper triangular})}$$

Gaussian elimination is indeed a triangular ~~triangulaziti~~ triangularization.

$$A \longrightarrow \underbrace{L_1}_{\text{lower triangular}} A = \begin{bmatrix} \times & \times & \cdots & \times \\ 0 & \times & & \times \\ \vdots & \vdots & & \\ 0 & \times & & \times \end{bmatrix}$$

$$\longrightarrow \underbrace{L_2}_{\text{lower triangula}} L_1 A = \begin{bmatrix} \times & \times & \times & \cdots & \times \\ 0 & \times & \times & & \times \\ 0 & 0 & \times & & \times \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & \times & & \times \end{bmatrix}$$

$$\longrightarrow L_{n-1} \cdots L_2 L_1 A = \underbrace{\begin{bmatrix} \times & \times & & \times \\ & \times & & \times \\ & 0 & \ddots & \vdots \\ & & & \times \end{bmatrix}}_{U \ (\text{upper triangula})}$$

$L_1, L_2, \dots, L_{n-1}$ are lower triangular matrices.

Product of lower triangular matrices is lower triangular.

Inverse of a lower triangular matrix is lower triangular.

Consequently as a by-product an LU factorization of the form

$$A = \underbrace{\left(L_1^{-1} L_2^{-1} \dots L_{n-1}^{-1}\right)}_{L} U$$

is computed.

$L \in \mathbb{C}^{n \times n}$: lower triangular

$U \in \mathbb{C}^{n \times n}$: upper triangular

EXAMPLE

$\boxed{\text{MULTIPLIERS}}$

$l_{21} = \frac{a_{21}}{a_{11}} = -2$

$l_{31} = \frac{a_{31}}{a_{11}} = 3$

$\boxed{\text{1ST COLUMN}}$

$$\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix} \xrightarrow[r_3 := r_3 - 3r_1]{r_2 := r_2 + 2r_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 5 & 8 \end{bmatrix}$$

is equivalent to

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 5 & 8 \end{bmatrix}$$

2nd column

$\boxed{\text{MULTIPLIER}}$ $l_{32} = \frac{a_{32}}{a_{22}} = 1$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 5 & 8 \end{bmatrix} \xrightarrow[r_3 := r_3 - r_2]{} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 15 \end{bmatrix}$$

is equivalent to

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{L_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 5 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 15 \end{bmatrix}}_{U}$$

Consequently

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 15 \end{bmatrix}}_{U}$$

$$\therefore \Longrightarrow$$

$$\underbrace{\begin{bmatrix} 1 & 2 & -2 \\ -2 & 1 & -3 \\ 3 & 11 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}_{L_1^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{L_2^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 15 \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}}_{L := L_1^{-1} L_2^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -7 \\ 0 & 0 & 15 \end{bmatrix}}_{U}$$

MULTIPLIERS $\left(l_{21}, l_{31}, l_{32} \right.$