LECTURE 19

LU FACTORIZATION WITH PARTIAL PIVOTING

Consider

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix}$$

with $\kappa(A) = \|A\|_2 \, \|A^{-1}\|_2 \approx 2.618$

Compute LU factorization

$$\begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10^{-20} & 1 \\ \boxed{10^{20}} & 1-10^{20} \end{bmatrix}$$

Exact factors

$$L = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 10^{-20} & 1 \\ 0 & 1-10^{20} \end{bmatrix}$$

Computed factors

$$\hat{L} = \begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix} \quad \text{and} \quad \hat{U} = \begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}$$

The backward error

$$\Delta A = A - \hat{L}\hat{U}$$

$$= \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

is significant.

CAUSE OF THE BACKWARD ERROR

* Large multiplier even though A is well-conditioned.

REMARK

* For numerical accuracy large multipliers must be avoided.

Suppose we swap the rows of A

$$\tilde{A} = \begin{bmatrix} 1 & 1 \\ 10^{-20} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ \boxed{10^{-20}} & 1-10^{-20} \end{bmatrix}$$

Computed factors

$$\hat{L} = \begin{bmatrix} 1 & 0 \\ 10^{-20} & 1 \end{bmatrix} \text{ and } \hat{U} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Backward error is small

$$\Delta A = \tilde{A} - \hat{L}\hat{U} = \begin{bmatrix} 0 & 0 \\ 0 & -10^{-20} \end{bmatrix}$$

OUTLINE OF THE PARTIAL PIVOTING STRATEGY

* Proceed column by column from left to right
* When processing the kth column
  - (i) Swap the rows j and k where j is such that

    $$|a_{jk}^{(k)}| = \max_{k \leqslant i \leqslant n} |a_{ik}^{(k)}| \quad \left(\text{LARGEST ENTRY ON THE KTH COLUMN}\right)$$

  - (ii) Make all the entries on the kth column below the diagonal zero (by applying row-replace operations)
  - (iii) The multipliers are stored and swapped together with the rows.

REMARK

Step (i) above ensures that

$$|m_{jk}| \leqslant 1$$

for $k = 1, \ldots, n-1$ and $j = k+1, \ldots, n$

EXAMPLE

$$\underbrace{\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix}}_{A^{(1)}} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{r_2 := r_2 - \frac{2}{3} r_1 \\ r_3 := r_3 - \frac{1}{3} r_1}]{} \underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 2/3 & 1/3 & 2 \\ 1/3 & 14/3 & 1 \end{bmatrix}}_{A^{(2)}}$$

$$\xrightarrow[r_2 \leftrightarrow r_3]{} \begin{bmatrix} 3 & 1 & 3 \\ 1/3 & 14/3 & 1 \\ 2/3 & 1/3 & 2 \end{bmatrix}$$

$$\xrightarrow[r_3 := r_3 - \frac{1}{14} r_2]{} \begin{bmatrix} 3 & 1 & 3 \\ 1/3 & 14/3 & 1 \\ 2/3 & 1/14 & \frac{27}{14} \end{bmatrix}$$

LU factorization for a row-permuted matrix

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/14 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 0 & 14/3 & 1 \\ 0 & 0 & 27/14 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 2 \\ 2 & 1 & 4 \end{bmatrix}}_{\tilde{A}}$$

where

$$\underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 2 \\ 2 & 1 & 4 \end{bmatrix}}_{\tilde{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_2} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix}}_{A}$$

Permutation Matrices

P_1 and P_2 above are permutation matrices.

* $P_1 A$ swaps the first and third rows of A
* $P_2 (P_1 A)$ swaps the second and third rows of $(P_1 A)$.

DEFN (Permutation Matrix)

A matrix $P \in \mathbb{R}^{n \times n}$ is called a <u>permutation matrix</u> if it has only one non-zero entry, that is equal to one, along each column and row.

- Left multiplication with a permutation matrix permutes rows

e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Right multiplication with a permutation matrix permutes columns

e.g. $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

<u>PROPERTIES</u>

(i) P_1, P_2 are permutation matrices $\Longrightarrow$ $P_1 P_2$ is a permutation matrix

(ii) $P^T P = I_n$ (P is orthogonal)

(iii) $P^2 = I_n$ (Since $P^T = P$)

Computed LU factorization with partial pivoting satisfies

$$\underbrace{P_{n-1} \cdots\cdots P_2 P_1}_{P} A = LU$$

* P_i is the permutation corresponding to the (row) swap when processing the ith column
* $P = P_{n-1} \cdots P_1$ is also a permutation matrix.

<u>Algorithm</u> (LU factorization with partial pivoting)

* Given $A \in \mathbb{C}^{n \times n}$ (non-singular)
* Produce an LU factorization of the form $PA = LU$ where P is a permutation matrix

for $k = 1, \ldots, n-1$
    Let l be s.t. $|a_{lk}| = \max_{k \leq i \leq n} |a_{ik}|$
    $\bar{a}_l \longleftrightarrow \bar{a}_k$, $\quad p_k = l$
    for $j = k+1, \ldots, n$
        $m_{jk} = a_{jk} / a_{kk}$
        $a_{j,k+1:n} = a_{j,k+1:n} - m_{jk} \, a_{k,k+1:n}$
        $a_{jk} = m_{jk}$
    end
end

REMARKS

* At termination
  (i) lower triangular portion of L is given by the corresponding portion of A.

  (ii) upper triangular and diagonal portion of U are given by the corresponding portion of A.

* The permutation matrix P is implicitly stored

$$p_k = l \iff P_k \in \mathbb{C}^{n \times n} \text{ swaps rows } k \text{ and } l.$$

* Total Flop Count is $\frac{2n^3}{3} + O(n^2)$

General Procedure (To solve $Ax=b$)	# of FLOPS
(i) Compute the factorization $PA = LU$	$\frac{2n^3}{3} + O(n^2)$
(ii) Note $PAx = LUx = Pb$. Form $\hat{b} := Pb$ (permute entries of b)	NO FLOPS (n-1 scalar swaps)
(iii) Define $\hat{x} = Ux$. Solve $L\hat{x} = \hat{b}$ by forward substitution	n^2
(iv) Solve $Ux = \hat{x}$ by back substitution	n^2