MATH 504
FALL 2011

LECTURE 19

LU FACTORIZATION

Gaussian elimination

$$A = A^{(1)} \xrightarrow[L_1]{} A^{(2)} \xrightarrow[L_2]{} A^{(3)} \cdots A^{(n-1)} \xrightarrow[L_{n-1}]{} A^{(n)} = \underbrace{U}_{\text{UPPER TRIANGULAR}}$$

$A^{(j)}$: After first j columns of A are processed. (Zeros are introduced below a_{kk} $k=1,\dots,j-1$.)

Closer look at the jth step

$$L_j \begin{bmatrix} \times & \cdots & \times & \times & \cdots & \times \\ & \ddots & & & & \\ & & \times & \times & \cdots & \times \\ & 0 & & a_{jj}^{(j)} & \cdots & \times \\ & & & \vdots & & \\ & & & a_{nj} & \cdots & \times \end{bmatrix}$$

$$= \begin{bmatrix} \times & \cdots & \times & \times & \times & \cdots & \times \\ & \ddots & & & & & \\ & & \times & \times & \times & \cdots & \times \\ & 0 & & a_{jj}^{(j)} & \times & \cdots & \times \\ & & & 0 & \times & & \times \\ & & & \vdots & \vdots & & \vdots \\ & & & 0 & \times & & \times \end{bmatrix}$$

where

$$L_j = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & 0 & & 1 & & \\ & & & -\ell_{(j+1)j} & \ddots & \\ & & & \vdots & & \\ & & & -\ell_{nj} & & 1 \end{bmatrix}$$

$$= I_n - \ell_j e_j^*$$

with

$$\ell_{kj} := a_{kj}^{(j)} / a_{jj}^{(j)} \qquad k = j+1, \dots, n$$

- $\ell_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \ell_{(j+1)j} \\ \vdots \\ \ell_{nj} \end{bmatrix}$ jth entry

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CLAINS

① $L_j^{-1} = (I_n + \ell_j e_j^*)$

PROOF

$$(I_n - \ell_j e_j^*)(I_n + \ell_j e_j^*)$$

$$= I_n - \ell_j \underbrace{e_j^* \ell_j}_{0} e_j^* = I_n$$

NOTE
$e_j^* \ell_j = 0$
because
jth entry
of ℓ_j is zero

② $L_j^{-1} L_{j+1}^{-1} = (I_n + l_j e_j^* + l_{j+1} e_{j+1}^*)$

PROOF

$$(I_n + l_j e_j^*)(I_n + l_{j+1} e_{j+1}^*)$$
$$= I_n + l_j e_j^* + l_{j+1} e_{j+1}^* + l_j \underbrace{e_j^* l_{j+1}}_{0} e_{j+1}$$
$$= I_n + l_j e_j^* + l_{j+1} e_{j+1}^*$$

(i.e. jth entry of l_{j+1} is zero)

③ More generally

$$L_1^{-1} \ldots L_{n-1}^{-1} = I_n + \sum_{j=1}^{n-1} l_j e_j^*$$

SUMMARY (LU Factorization)

$$A = \underbrace{(L_1^{-1} \ldots L_{n-1}^{-1})}_{L} U$$

where

$$L = \begin{bmatrix} 1 & & & & 0 \\ l_{21} & \ddots & & & \\ \vdots & & \ddots & & \\ l_{(n-1)1} & & & 1 & \\ l_{n1} & \cdots & \cdots & l_{n(n-1)} & 1 \end{bmatrix}$$

with

$$l_{kj} = a_{kj}^{(j)} / a_{jj}^{(j)}$$

Algorithm (LU Factorization)

Input : $A \in \mathbb{C}^{n\times n}$

Output : $L \in \mathbb{C}^{n\times n}$ (lower triangular) and
$U \in \mathbb{C}^{n\times n}$ (upper triangular) s.t.
$A = LU$.

$L = I_n$

for $j = 1, \ldots, n-1$ (j : col to be processed)

  for $k = j+1, \ldots, n$

  % Introduce zero at a_{kj}

    $\left. l_{kj} = a_{kj} / a_{jj} \right\}$ 1 FLOP

    $\left. a_{k,j:n} = a_{k,j:n} - l_{kj} \cdot a_{j,j:n} \right\}$ $2(n-j+1)$ FLOPS

  end

end

$U = A$

$$\text{TOTAL \# FLOPS} = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} 2(n-j) + 3$$

$$= \sum_{j=1}^{n-1} 2(n-j)^2 + 3(n-j)$$

$$= \sum_{j=1}^{n-1} 2j^2 + 3j$$

$$= 2\left(\frac{(n-1)\cdot n\cdot(2n-1)}{6}\right) + 3(n-1)$$

$$= \frac{2n^3}{3} + O(n^2)$$

SOLUTION OF LINEAR SYSTEMS USING LU FACTORIZATION

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}}_{L} \underbrace{\underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\hat{x}} = \begin{bmatrix} 9 \\ -5 \\ 11 \end{bmatrix}$$

(1) Forward Stage

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 11 \end{bmatrix} \Longrightarrow \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$$

FORWARD SUBSTITUTE

(2) Backward Stage

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

BACK SUBSTITUTE

PROCEDURE (To solve $Ax=b$)

	# FLOPS
(1) Compute the LU Factorization $$A = LU$$	$\frac{2n^3}{3} + O(n^2)$
(2) Let $\hat{x} := Ux$. Solve $$L\hat{x} = b$$ by forward substitution	n^2
(3) Solve $$Ux = \hat{x}$$ by back substitution	n^2

GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING (GEPP)

$$A = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix} \qquad \kappa(A) = \|A\|_2 \, \|A^{-1}\|_2 \approx 2.618$$

Compute LU Factorization

$$\begin{bmatrix} 10^{-20} & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10^{-20} & 1 \\ \boxed{10^{20}} & 1-10^{20} \end{bmatrix}$$

Exact LU Factorization

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20}+1 \end{bmatrix}}_{U}$$

Computed LU Factorization

$$\tilde{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 10^{20} & 1 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{bmatrix}}_{\tilde{U}} = \begin{bmatrix} 10^{-20} & 1 \\ 1 & 0 \end{bmatrix}$$

Backward Error

- $$A - \tilde{L}\tilde{U} = \Delta A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Cause of large backward error

* Large multipliers ($l_{21} = 10^{20}$; rounding errors as large as $10^{20}\epsilon_{mach}$)

Now swap the rows of A

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 10^{-20} & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ \hline 10^{-20} & 1-10^{-20} \end{bmatrix}$$

Computed LU Factorization

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 10^{-20} & 1 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\tilde{U}} = \begin{bmatrix} 1 & 1 \\ 10^{-20} & 1+10^{-20} \end{bmatrix}$$

Backward Error

$$\Delta A = \hat{A} - \tilde{L}\hat{U} = \begin{bmatrix} 0 & 0 \\ 0 & 10^{-20} \end{bmatrix}$$

REMARK

Large multipliers must be avoided for backward stability.

Outline (GEPP)

* Proceed column by column from left to right
* When processing jth column

(1) Find k s.t.

$$|a_{kj}^{(j)}| = \max_{j \leqslant l \leqslant n} |a_{lj}^{(j)}|$$

(2) Swap columns k and j together with multipliers corresponding to these rows

(3) Make all entries below diagonal zero on the jth column by applying row-replace operations.

REMARK

Swapping the columns ensure that

$$|l_{kj}| \leqslant 1$$

for $k = j+1, \dots, n$ (and $j = 1, \dots, n$)

EXAMPLE

$$\begin{bmatrix} 1 & 5 & 2 \\ 2 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_3]{①} \begin{bmatrix} 3 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 5 & 2 \end{bmatrix}$$

$$\xrightarrow[\substack{r_2 := r_2 - \frac{2}{3} r_1 \\ r_3 := r_3 - \frac{1}{3} r_1}]{} \begin{bmatrix} 3 & 1 & 3 \\ 2/3 & 1/3 & 2 \\ 1/3 & 14/3 & 1 \end{bmatrix}$$

$$\xrightarrow[r_2 \leftrightarrow r_3]{②} \begin{bmatrix} 3 & 1 & 3 \\ 1/3 & 14/3 & 1 \\ 2/3 & 1/3 & 2 \end{bmatrix}$$

$$\xrightarrow[r_3 := r_3 - \frac{1}{14} r_2]{} \begin{bmatrix} 3 & 1 & 3 \\ 1/3 & 14/3 & 1 \\ 2/3 & 1/14 & 27/14 \end{bmatrix}$$

If rows 1 and 3, then rows 2 and 3 were initially swapped, then an LU factorization would be computed (without steps ① and ②)

Consequently

$$\underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 1 & 5 & 2 \\ 2 & 1 & 4 \end{bmatrix}}_{\hat{A}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 2/3 & 1/14 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 1 & 3 \\ 0 & 14/3 & 1 \\ 0 & 0 & 27/14 \end{bmatrix}}_{U}$$

An LU factorization is calculated for a row-permuted version of A, i.e.

$$\hat{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\substack{P_2 \\ \text{permutation matrix} \\ \text{corresponding to (2)}}} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\substack{P_1 \\ \text{permutation matrix corresponding} \\ \text{to (1)}}} A$$