MATH 504
FALL 2011

LECTURE 1

LINEAR SYSTEMS

Given $A \in \mathbb{C}^{n\times n}$ and $b \in \mathbb{C}^n$.

Find an $x \in \mathbb{C}^n$ satisfying $Ax = b$.

Some applications

* Numerical solution of nonlinear systems
* Numerical solution of optimization problems
* Numerical solution of differential equations

EIGENVALUE PROBLEMS

Given $A \in \mathbb{C}^{n\times n}$.

Find pair (x, λ) where $x \neq 0 \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ satisfying $Ax = \lambda x$.

Some applications

* Analytical solution of linear differential equations
* Analysis of dynamical systems, for instance issues such as stability, resonance.

OPERATION COUNT

The criterion used for the efficiency of an algorithm is the number of <u>flops</u> performed by the algorithm.

FLOPS (Floating Point Operations)

$\oplus, \ominus, \otimes, \oslash$

addition
on a computer

Flop Count is crude

* The costs of $\oplus, \ominus, \otimes, \oslash$ are not the same in reality.
* The data transfers (take significant amount of time) are ignored.

<u>CASE STUDY</u> (Matrix-Matrix Product)

Let $A \in \mathbb{C}^{n \times p}$ and $B \in \mathbb{C}^{p \times m}$.

$$X := A \cdot B \in \mathbb{C}^{n \times m}$$

Recall

$$x_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj} \quad \left(\begin{array}{l} \text{PRODUCT OF} \\ i\text{th ROW OF } A \\ \text{AND } j\text{th COLUMN OF } B \end{array} \right)$$

Algorithm (Matrix-Matrix Product)

Input: $A \in \mathbb{C}^{n \times p}$, $B \in \mathbb{C}^{p \times m}$

Output: $X = A \cdot B \in \mathbb{C}^{n \times m}$

For $i = 1, \dots, n$

  For $j = 1, \dots, m$

  (Form the entry x_{ij} here)

    For $k = 1, \dots, p$

      $x_{ij} = x_{ij} + a_{ik} b_{kj}$ } 2 FLOPS

    End

  End

End

For each (i,j,k) 2 Flops are performed.

$$\text{Total \# of Flops} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{p} 2$$

$$= 2\, n \cdot m \cdot p$$

Specifically if $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$

Total # of Flops $= 2n^3$

O-NOTATIONS

Big-O notation

Let f, g be univariate and scalar-valued functions.

$f(x) = O(g(x))$ as $x \to \infty$ means

There exist scalars c and ε s.t.

$$|f(x)| \leqslant c|g(x)| \quad \forall x > \varepsilon$$

($g(x)$ grows asymptotically at least as fast as $f(x)$)

More generally $f(x) = O(g(x))$ as $x \to a$ means

There exist scalars c and ε s.t.

$$|f(x)| \leqslant c|g(x)| \quad \forall x \in (a-\varepsilon, a+\varepsilon)$$

EXAMPLE

For matrix-matrix product (with square A and B)

$$\text{Total \# of Flops} = 2n^3$$
$$= O(n^3)$$

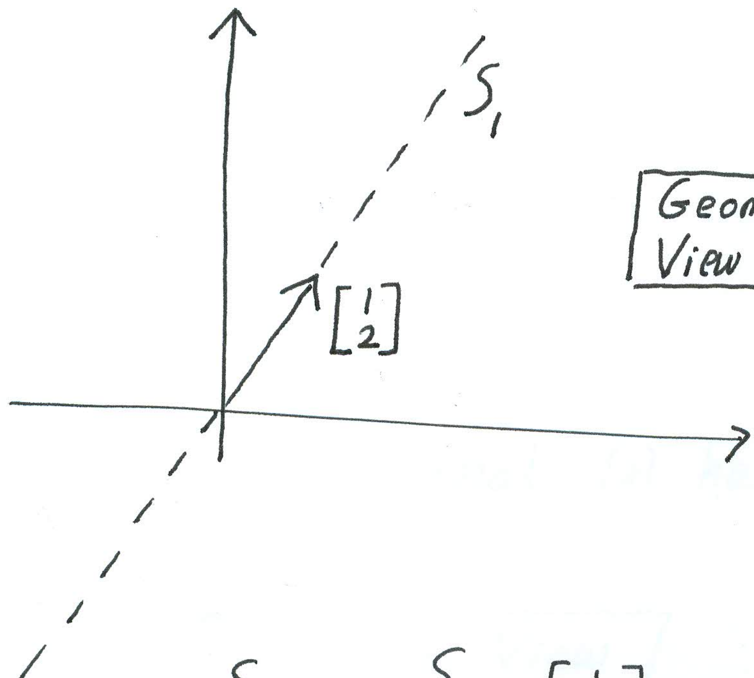
* $2n^3 = O(n^4)$ also, but $2n^3 \neq O(n^2)$.

REMARK

For our purposes if the limit $x \to a$ is not specified as the example above, the convention is that the limit is taken as $x \to \infty$.

REVIEW ON VECTOR SPACES

$$S_1 := \left\{ c\begin{bmatrix}1\\2\end{bmatrix} : c \in \mathbb{R} \right\}$$
$$= \operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix} \right\}$$



$\boxed{\text{Geometric View of } S_1}$ line with slope 2 going through the origin

$$S_2 := \left\{ c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$
$$= \operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}$$

Claim

$$S_2 = \mathbb{R}^2$$

Proof

Let $b \in \mathbb{R}^2$. Need to show the existence of c_1 and c_2 s.t.

$$(*) \quad c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = b$$

$$\iff \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = b \quad \text{has a soln. } c_1, c_2$$

Row reduction

$$\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 1 & b_2 \end{bmatrix} \xrightarrow[r_2 := r_2 - 2r_1]{} \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -3 & b_2 - 2b_1 \end{bmatrix}$$

$$c_2 = \frac{b_2 - 2b_1}{(-3)} \qquad \begin{aligned} c_1 &= b_1 - 2c_2 \\ &= b_1 + \tfrac{2}{3}(b_2 - 2b_1) \end{aligned}$$

This proves that (*) has a soln for all b.

$\square$

$\boxed{\text{Geometric View of } S_2}$ whole space $\mathbb{R}^2$

<u>DEFN</u> (Linear Combination & Span)

Let $v_1, \ldots, v_m \in \mathbb{R}^n$.

(1) Any sum

$$c_1 v_1 + c_2 v_2 + \ldots + c_m v_m \in \mathbb{R}^n$$

where $c_1, \ldots, c_m \in \mathbb{R}$ is called a <u>linear combination</u>.

(2)

$$\operatorname{span}\{v_1, \ldots, v_m\} := \{c_1 v_1 + \ldots + c_m v_m : c_1, \ldots, c_m \in \mathbb{R}\} \subseteq \mathbb{R}^n$$

OBSERVATIONS

(1) S_1 and S_2 are closed under addition, i.e.,

$$v_1, v_2 \in S_1 \implies v_1 + v_2 \in S_1$$

(2) S_1 and S_2 are closed under scalar multiplication, i.e.,

$$c \in \mathbb{R},\ v \in S_1 \implies cv \in S_1$$

In general span of m vectors in $\mathbb{R}^n$ is a hyper-plane (a linear object satisfying observations (1) and (2)) containing the origin.

There are other sets that are closed under addition and scalar multiplication.

EXAMPLES

① $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$

$\mathbb{P}_2$ is closed under addition

$$(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in \mathbb{P}_2$$

$\mathbb{P}_2$ is closed under scalor multiplication

$$c(a_0 + a_1 x + a_2 x^2) = ca_0 + ca_1 x + c_2 a_2 x^2 \in \mathbb{P}_2$$

$$\mathbb{R}^{2\times 2} = \left\{ \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix} : a_0, a_1, a_2, a_3 \in \mathbb{R} \right\}$$

DEFN (Vector Space)

A set V, over which an addition and a scalar multiplication operations are defined, is called a vector space if

(1) $\forall v_1, v_2 \in V \quad v_1 + v_2 \in V$, and

(2) $\forall v \in V, \forall c \in \mathbb{R} \quad cv \in V$

(3) Addition and scalar multiplication satisfy certain properties such as commutativity, associativity.

$\mathbb{R}^n$, $\mathbb{P}_2$ and $\mathbb{R}^{n\times n}$ with usual additions and scalar multiplications are vector spaces.

$$\mathbb{B}_1 = \{ v \in \mathbb{R}^n : v^T v \leq 1 \} \quad \left(\begin{array}{c} \text{CLOSED} \\ \text{UNIT} \\ \text{BALL} \end{array} \right)$$

is not a vector space

* Not Closed under addition
* Not Closed under scalar multiplication

$S_1 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is a subset of $\mathbb{R}^2$ that is closed under addition and scalar multiplication.

We say S_1 is a <u>subspace</u> of $\mathbb{R}^2$

<u>DEFN</u> (Subspace)

A subset S of a vector space V is called a subspace of V if

(1) S is closed under addition, and

(2) S is closed under scalar multiplication.

EXAMPLES

① $\{a_0 + a_2x^2 : a_0, a_2 \in \mathbb{R}\}$

is a subspace of $\mathbb{P}_2$

② $\left\{\begin{bmatrix}a_0 & a_1\\0 & a_2\end{bmatrix} : a_0, a_1, a_2 \in \mathbb{R}\right\}$

is a subspace of $\mathbb{R}^{2\times 2}$