LECTURE 2

ORTHOGONALITY – BASIC DEFINITIONS

Complex Numbers

Let $\underset{\text{set of complex numbers}}{z \in \mathbb{C}}$. Then $\bar{z}$ denotes the complex conjugate of z.

e.g.

$$z = 2-2i, \qquad \bar{z} = 2+2i$$

Polar Representation: Given $r, \theta \in \mathbb{R}$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{r+i\theta} = e^{r}(\cos\theta + i\sin\theta)$$

All complex numbers have polar representations. Given $z_r, z_i \in \mathbb{R}$

$$z = z_r + i z_i = \underbrace{\sqrt{z_r^2+z_i^2}}_{|z|}\left(\frac{z_r}{\sqrt{z_r^2+z_i^2}} + i\frac{z_i}{\sqrt{z_r^2+z_i^2}}\right)$$

$$z = |z|\, e^{i\theta}$$

where $\theta =$ signed $\arctan\left(\frac{z_i}{z_r}\right)$

e.g.

$$z = 2-2i = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\, e^{-\frac{\pi}{4}i}$$

Basic properties of complex conjugation

(1) Let $z = |z| e^{i\theta}$. Then $\bar{z} = |z| e^{-i\theta}$

(2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Inner Products

A function $\langle \cdot , \cdot \rangle$ that takes two vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) as input and maps them to a scalar.

$$\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \quad \left(\begin{array}{c}\text{real}\\ \text{inner}\\ \text{product}\end{array}\right)$$

$$\mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C} \quad \left(\begin{array}{c}\text{complex}\\ \text{inner}\\ \text{product}\end{array}\right)$$

Standard Inner Products (Dot products)

in $\mathbb{R}^n$: $\langle u, v \rangle = u \cdot v$

$$= u^T v$$

e.g. $\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$= (1)(1) + (2)(-1) + (3)(1) = \underline{\underline{2}}$$

in $\mathbb{C}^n$: $\langle u, v \rangle = u \cdot v$

$$= u^* v$$

u^* : complex conjugate transpose of u

e.g. $\left\langle \begin{bmatrix} 1+2i \\ -i \end{bmatrix}, \begin{bmatrix} 2 \\ 3+4i \end{bmatrix} \right\rangle$

$$=$$

$$\begin{bmatrix} 1-2i & i \end{bmatrix} \begin{bmatrix} 2 \\ 3+4i \end{bmatrix}$$

$$=$$

$$2(1-2i) + (3+4i)i = -2 - i$$

<u>Orthogonality</u>

We say $u, v \in \mathbb{R}^n$ (or $\mathbb{C}^n$) are orthogonal if $\langle u, v \rangle = 0$.

e.g.

In $\mathbb{R}^2$ $\quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\qquad u \perp v$

are orthogonal

$(\langle u, v \rangle = u^T v = 0)$

$u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad u \not\perp v.$

are not orthogonal

$(\langle u, v \rangle = u^T v = 3)$

Orthogonal sets: A set of vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called orthogonal if each pair of vectors in the set are orthogonal

$U = \{u_1, \dots, u_k\}$ is an orthogonal set $\iff$ $\langle u_i, u_j \rangle = 0$ for all i, j s.t. $i \neq j$

e.g. $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{u_3} \right\}$ is an orthogonal set in $\mathbb{R}^3$

$$\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0$$

$\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}}_{v_3} \right\}$ is not orthogonal

$$\langle v_1, v_3 \rangle = 4 \neq 0$$

Unitary Matrices

$A \in \mathbb{C}^{n \times n}$ is called unitary if

$$A^* A = \underbrace{I}_{\substack{n\times n \\ \text{identity}}} \quad \left(\text{that is } A^{-1} = A^* \right)$$

A^* : Complex conjugate transpose of the matrix A

e.g.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \text{ is unitary}$$

since $$\underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}}_{A^*} \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Special case when A is real

$A \in \mathbb{R}^{n\times n}$ is called orthogonal if

$$A^T A = I \quad \left(\text{that is } A^{-1} = A^T\right)$$

e.g. $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is orthogonal

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & +1 \\ -1 & 1 \end{bmatrix}}_{A^T} \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthonormal sets: A set of vectors in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is called orthonormal if the set is

(1) orthogonal,

(2) consists of vectors of unit length.

e.g.

$$\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} \text{ is orthonormal}$$

<u>REMARK</u>

The columns of a unitary (or orthogonal) matrix form an orthonormal set.

$$A^* A = I \iff \begin{array}{l} a_i^* a_j = 0 \quad i \neq j \\ \underbrace{a_i^* a_i = 1}_{\text{length of } a_i \text{ is one}} \end{array}$$

$\iff \{a_1, \dots, a_n\}$ is orthonormal

Note that

$$\text{Euclidean length of } u \in \mathbb{C}^n = \sqrt{|u_1|^2 + \dots + |u_n|^2}$$

$$= \sqrt{\bar{u}_1 u_1 + \bar{u}_2 u_2 + \dots + \bar{u}_n u_n}$$

$$= \sqrt{u^* u} = \sqrt{\langle u, u \rangle}$$

$$\left(\sqrt{u^T u} \text{ if } u \in \mathbb{R}^n \right)$$

VECTOR NORMS

<u>DEFN</u>

A vector norm $\|\cdot\|$ is a function from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}$ satisfying

POSITIVITY (i) $\|v\| > 0$ for all nonzero $v \in \mathbb{R}^n$ (or $\mathbb{C}^n$)

HOMOGENEITY (ii) $\|\alpha v\| = |\alpha| \|v\|$ for all $\alpha \in \mathbb{R}$, $v \in \mathbb{R}^n$ (or $\alpha \in \mathbb{C}$, $v \in \mathbb{C}^n$)

TRIANGULAR INEQUALITY (iii) $\|v + w\| \leqslant \|v\| + \|w\|$ for all $v, w \in \mathbb{R}^n$ (or $v, w \in \mathbb{C}^n$)

2-norm (Euclidean length): $\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2}$

1-norm : $\|v\|_2 = \sum_{i=1}^{n} |v_i|$

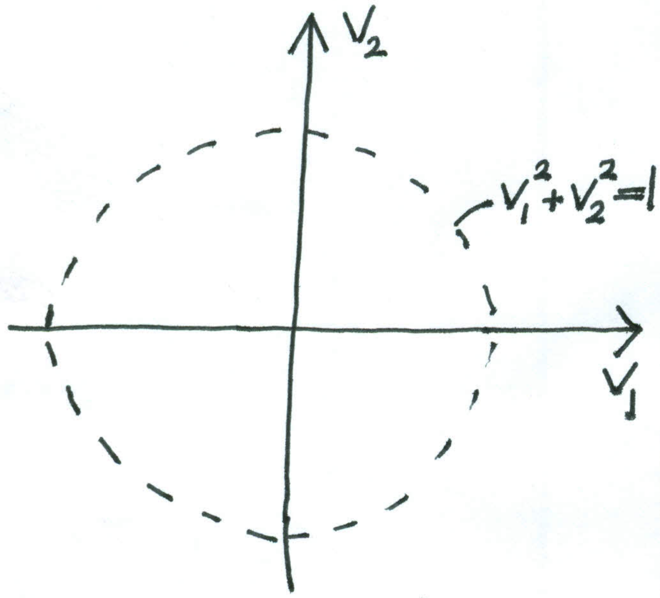
∞-norm : $\|v\|_\infty = \max_{1 \leqslant i \leqslant n} |v_i|$

2, 1, ∞ − norms are special cases of p-norm

$$\|v\|_p = \sqrt[p]{\sum_{i=1}^{n} |v_i|^p} \qquad p \geqslant 1$$

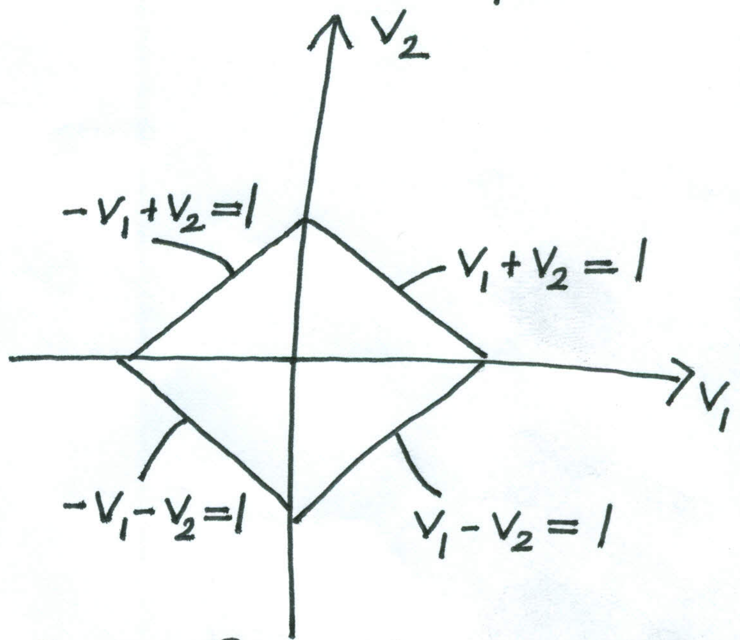
$$S_{\|\cdot\|} = \{ v \in \mathbb{R}^2 : \|v\|_2 = 1 \}$$

$\|\cdot\| = \|\cdot\|_2$



$$S_{\|\cdot\|_2} = \{ v \in \mathbb{R}^2 : v_1^2 + v_2^2 = 1 \}$$

$\|\cdot\| = \|\cdot\|_1$



$$S_{\|\cdot\|_1} = \{ v \in \mathbb{R}^2 : |v_1| + |v_2| = 1 \}$$

<u>THM</u> (Hölder's inequality)

Suppose $p, q \geqslant 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then for all $x, y \in \mathbb{C}^n$

$$|x^* y| \leq \|x\|_p \|y\|_q$$

<u>COROLLARY</u> (Schwarz inequality)

$$|x^* y| \leqslant \|x\|_2 \|y\|_2$$

REMARK

Unitary transformations preserve the Euclidean-length (2-norm) of vectors, i.e.

$$U \in \mathbb{C}^{n\times n} \text{ is unitary}$$

$$\Longrightarrow$$

$$\|U^* x\|_2 = \|x\|_2 \quad \text{for all } x \in \mathbb{C}^n$$

$$\left(\text{since} \quad \|Ux\|_2 = \sqrt{(Ux)^* Ux} = \sqrt{x^* U^* U x} = \sqrt{x^* x} = \|x\|_2 \right)$$

$\boxed{\text{MATRIX NORMS}}$

DEFN

A matrix norm $\|\cdot\|$ is a function from $\mathbb{C}^{n\times n}$ to $\mathbb{R}$ satisfying

(i) $\|A\| > 0$ for all nonzero $A \in \mathbb{C}^{n\times n}$

(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for all $\alpha \in \mathbb{C}$ and $A \in \mathbb{C}^{n\times n}$

(iii) $\|A+B\| \leqslant \|A\| + \|B\|$ for all $A, B \in \mathbb{C}^{n\times n}$

Some common matrix norms

$\boxed{\text{FROBENIUS NORM}}$

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\sum_{i=1}^{n} \|a_i\|^2} = \sqrt{\sum_{i=1}^{n} a_i^* a_i} = \sqrt{\operatorname{trace}(A^* A)}$$

e.g. $$\left\|\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\right\|_F = \sqrt{(2)^2+(-1)^2+(3)^2+(2)^2} = 3\sqrt{2}$$

Note that
$$\operatorname{trace}(A) = \sum_{i=1}^{n} a_{ii} \quad (\text{where } A \in \mathbb{C}^{n\times n})$$

$\boxed{\begin{array}{c}\text{2-NORM}\\ \text{(SPECTRAL NORM)}\end{array}}$

$$\|A\|_2 = \max_{\substack{x\in\mathbb{R}^n \\ \text{s.t. } \|x\|_2=1}} \|Ax\|_2$$

e.g.
$$\left\|\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\right\|_2 = \max_{\|x\|_2=1} \left\|\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\|_2$$
$$= \max_{\|x\|_2=1} \left\|\begin{bmatrix} 2x_1 - x_2 \\ x_1 + 2x_2 \end{bmatrix}\right\|_2$$
$$= \sqrt{(2x_1-x_2)^2 + (x_1+2x_2)^2}$$
$$= \sqrt{5x_1^2+5x_2^2} = \sqrt{5}\sqrt{x_1^2+x_2^2} = \sqrt{5}$$

$\boxed{\begin{array}{c}\text{1-NORM}\\ \text{(MAXIMAL COLUMN SUM)}\end{array}}$

$$\|A\|_1 = \max_{\substack{x\in\mathbb{R}^n \\ \text{s.t. } \|x\|_2=1}} \|Ax\|_1$$

THM

$$\|A\|_1 = \max_{1 \leqslant i \leqslant n} \|a_i\|_1$$

PROOF

Let $x \in \mathbb{R}^n$ such that $\|x\|_1 = 1$

$$\|Ax\|_1 = \|x_1 a_1 + x_2 a_2 + \ldots\ldots + x_n a_n\|_1$$

$$\leqslant \|x_1 a_1\|_1 + \ldots + \|x_n a_n\|_1 \quad \left(\begin{array}{c}\text{By triangular} \\ \text{inequality}\end{array}\right)$$

$$= |x_1|\,\|a_1\|_1 + \ldots + |x_n|\,\|a_n\|_1 \quad \left(\begin{array}{c}\text{By} \\ \text{homogeneity}\end{array}\right)$$

Consider j s.t. $\|a_j\|_1 = \max_{1 \leqslant i \leqslant n} \|a_i\|_1$. Then

$$\|Ax\|_1 \leqslant \|a_j\|_1 \underbrace{(|x_1| + \ldots\ldots + |x_n|)}_{\|x\|_1 = 1} = \|a_j\|_1$$

$$\Longrightarrow$$

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 \leqslant \|a_j\|_1$$

For $\tilde{x} = [0 \ldots 0 \; 1 \; \overset{j\text{th entry}}{0} \ldots 0]^T$ $(\|\tilde{x}\| = 1)$

$$\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 \geqslant \|A\tilde{x}\|_1 = \|a_j\|_1 \quad \square$$

e.g.

$$\left\| \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \right\|_1 = \max\left(\left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|_1, \left\| \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\|_1 \right) = 5$$

$\boxed{\begin{matrix}\infty\text{-NORM} \\ \text{(MAXIMAL ROW SUM)}\end{matrix}}$ $\quad \|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty$

THM

$$\|A\|_\infty = \max_{1\leqslant i\leqslant n} \|\bar{a}_i^T\|_1$$

e.g.

$$\left\|\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}\right\|_\infty = \max\begin{pmatrix} \|[2 \quad -3]^T\|_1, \\ \|[1 \quad 2]^T\|_1 \end{pmatrix} = 5$$

Vector induced matrix (p,q)-norm

$$\|A\|_{(p,q)} = \max_{\substack{x\in\mathbb{R}^n \\ \text{s.t. } \|x\|_q=1}} \|Ax\|_p$$

(1-norm, 2-norm, ∞-norm are special cases)

REMARK

Let $U \in \mathbb{C}^{n\times n}$ be a unitary matrix.

$$\|U\|_2 = \max_{\|x\|_2=1} \|Ux\|_2 = \max_{\|x\|_2=1} \|x\|_2$$
$$= 1$$

$$\|U\|_F = \sqrt{\text{trace}\,(U^*U)}$$
$$= \sqrt{\text{trace}\,(I)}$$
$$= \sqrt{n}$$

Furthermore for any $A \in \mathbb{C}^{n\times n}$

$$\|UA\|_2 = \max_{\|x\|_2=1} \|UAx\| = \max_{\|x\|_2=1} \|Ax\| = \|A\|_2$$

$$\|UA\|_F = \sqrt{\operatorname{trace}\left((UA)^* UA\right)} = \sqrt{\operatorname{trace}\left(A^* U^* UA\right)} = \sqrt{\operatorname{trace}(A^* A)} = \|A\|_F$$