LECTURE 20

GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING (GEPP)

$$A = A^{(1)} \xrightarrow[\substack{P_1 \text{ (Permute)} \\ L_1 \text{ (Row-replace)}}]{} A^{(2)} \xrightarrow[\substack{P_2 \\ L_2}]{} A^{(3)} \cdots \xrightarrow[\substack{P_{n-1} \\ L_{n-1}}]{} A^{(n)} = U$$

An LU Factorization of the form

$$\underbrace{(P_{n-1} \dots P_1)}_{P} A = \underbrace{(L_1^{-1} \dots L_{n-1}^{-1})}_{L} U$$

is computed.

Algorithm (LU Factor with Pivoting)

Input : $A \in \mathbb{C}^{n \times n}$

Output : $L \in \mathbb{C}^{n \times n}$ (lower triangular)

$U \in \mathbb{C}^{n \times n}$ (upper triangular)

~~and~~ $P \in \mathbb{R}^{n \times n}$ (permutation matrix)

s.t. $PA = LU$.

$L = I_n$

for $j = 1, \dots, (n-1)$ (j : col to be processed)

  Find ℓ s.t. $|a_{\ell j}| = \max_{j \leq k \leq n} |a_{kj}|$

  $a_{\ell, j:n} \longleftrightarrow a_{j, j:n}$ (swap rows ℓ and j)

  $p_j = \ell$

  $\ell_{\ell, 1:j-1} \longleftrightarrow \ell_{j, 1:j-1}$ (swap also multipliers)

  for $k = j+1, \dots, n$

    $\ell_{kj} = a_{kj} / a_{jj}$

    $a_{k, j:n} = a_{k, j:n} - \ell_{kj} a_{j, j:n}$

  end

end

$U = A$

REMARKS

* Permutation matrix P_j is implicitly kept.

  $p_j = \ell$ means P_j permutes rows j and ℓ

* TOTAL # FLOPS $= \frac{2n^3}{3} + O(n^2)$

Procedure (To solve $Ax = b$)

	# FLOPS
(1) Compute an LU Factorization $$PA = LU$$	$\frac{2n^3}{3} + O(n^2)$
(2) Note $PAx = Pb$. Calculate $$\hat{b} = Pb$$	NO FLOPS
(3) Let $\hat{x} := Ux$. Solve $$L\hat{x} = \hat{b}$$ by forward substitution	n^2
(4) Solve $$Ux = \hat{x}$$ by back substitution	n^2