Math 504 (Fall 2010) - Lecture 21

Eigenvalues, Basics

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Outline

- Eigenvalues, basic definitions and facts (Trefethen&Bau, Lecture 24)
- Eigenvalues, motivation

Eigenvalues, basic definitions and facts

Definition (Eigenvalues and Eigenvectors):

Let $A \in \mathbb{C}^{n\times n}$. Suppose that

$$Ax = \lambda x$$

for some scalar $\lambda \in \mathbb{C}$ and nonzero vector $x \in \mathbb{C}^n$. Then

(i) λ is called an eigenvalue of A, and

(ii) x is called an eigenvector of A associated with λ.

Eigenvalues, basic definitions and facts

Example:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1} = \underbrace{1}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2} = \underbrace{3}_{\lambda_2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2}$$

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$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors associated with λ_1 and λ_2.

Eigenvalues, basic definitions and facts

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$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \text{ for } x \neq 0 \\ &\iff Ax - \lambda x = (A - \lambda I)x = 0 \text{ for } x \neq 0 \\ &\iff A - \lambda I \text{ is singular} \end{aligned}$$

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$$\det(A - \lambda I) \quad = \quad \det\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Eigenvalues, basic definitions and facts

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$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\ &= \det\left(\begin{bmatrix} -1-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix} \right) \end{aligned}$$

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Eigenvalues of A

$\det(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$,

so the eigenvalues (the roots of $\det(A - \lambda I)$) are $\lambda_1 = -3$, $\lambda_2 = 1$.

Eigenvalues, basic definitions and facts

Characteristic Polynomial

$p(\lambda) = \det(A - \lambda I)$ is a monic polynomial of λ of degree n and called the *characteristic polynomial* of A.

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The eigenvalues of $A \in \mathbb{C}^{n \times n}$ are the roots of its characteristic polynomial.

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$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad \text{where } a_n \neq 0.$$

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$$p(z) = 0 \quad \Longleftrightarrow \quad \tilde{p}(z) = 0$$

Eigenvalues, basic definitions and facts

Theorem 2:

λ is a root of $\tilde{p}(z) = z^n + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0$

$$\Longleftrightarrow$$

λ is an eigenvalue of the $n \times n$ companion matrix

$$\mathcal{C} = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}$$

Eigenvalues, basic definitions and facts

Proof: Suppose $\tilde{p}(\lambda) = 0$. Then

$$\mathcal{C}\begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

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implying λ is a root of $\tilde{p}(z)$.

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  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).
- Need for iterative algorithms for eigenvalue computation
  - Only in the limit as the number of iterations go to ∞ the estimates approach eigenvalues.

Eigenvalues, basic definitions and facts

Theorem (Eigenvalues and Characteristic Polynomial)

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Definition (Algebraic Multiplicity)

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$. The multiplicity of λ as a root of $p(\lambda) = \det(A - \lambda I)$ is called the algebraic multiplicity of λ.

Eigenvalues, basic definitions and facts

Theorem (Calculation of Eigenvectors)

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$. Then v is an eigenvector associated with $\lambda \Longleftrightarrow (A - \lambda I)v = 0$ and $v \neq 0$.

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Example:

The matrix $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -3$, $\lambda_2 = 1$.

Eigenvalues, basic definitions and facts

Find an eigenvector v_1 associated with $\lambda_1 = -3$ (below $c \neq 0$)

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$$\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$

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Find an eigenvector v_1 associated with $\lambda_1 = -3$ (below $c \neq 0$)

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$$\Longrightarrow \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

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Finding an eigenvector v_2 associated with $\lambda_2 = 1$ (below $c \neq 0$)

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Finding an eigenvector v_2 associated with $\lambda_2 = 1$ (below $c \neq 0$)

$$\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_2 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$

$$\Longrightarrow \boxed{v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Eigenvalues, basic definitions and facts

Definition (Eigenspace):

Let λ be an eigenvalue of $A \in \mathbf{C}^{n \times n}$. The set $E_\lambda = \mathrm{Null}(A - \lambda I)$ is called the eigenspace of A associated with λ.

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that is $\{Ax : x \in E_\lambda\} \subseteq E_\lambda$.

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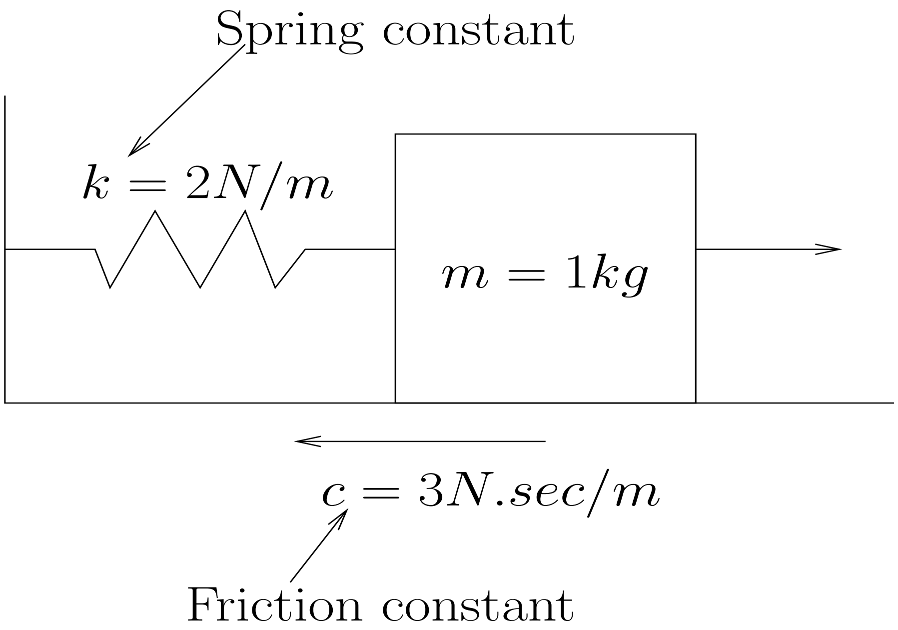
Definition (Geometric Multiplicity)

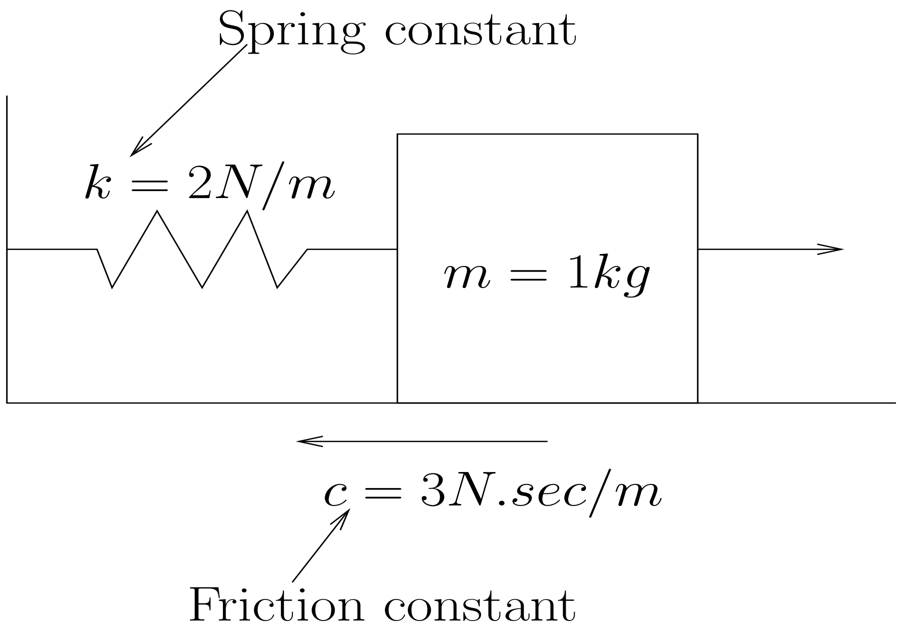
Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$. The dimension of the eigenspace $E_\lambda = \text{Null}(A - \lambda I)$ associated with λ is called the geometric multiplicity of λ.

Eigenvalues, motivation



The motion of vibrating structures is governed by eigenvalues.

Eigenvalues, motivation



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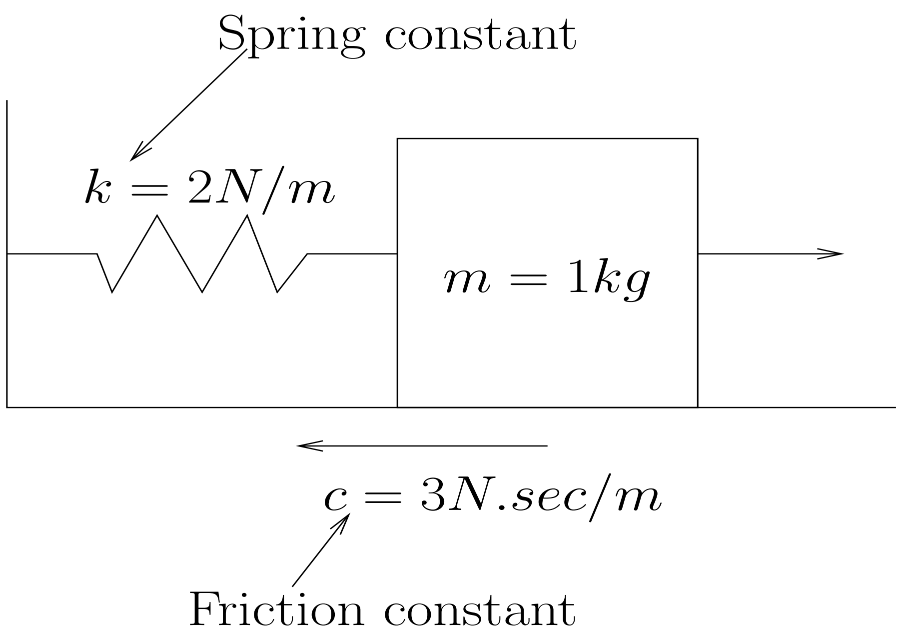
$$\text{Net Force} = ma(t)$$

Eigenvalues, motivation



The motion of vibrating structures is governed by eigenvalues.

- By Newton's law of motion

$$\text{Net Force} = ma(t)$$

- The friction and springs apply forces against displacement

$$\text{Net Force} = -c\ v(t) - k\ x(t)$$

Eigenvalues, motivation

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$$\begin{aligned} v'(t) + 3v(t) + 2x(t) &= 0 \\ -v(t) + x'(t) &= 0 \end{aligned}$$

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$$\underbrace{\begin{bmatrix} v'(t) \\ x'(t) \end{bmatrix}}_{y'(t)} = \underbrace{\begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v(t) \\ x(t) \end{bmatrix}}_{y(t)}$$

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$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$

with the associated eigenvectors $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

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$$c_k e^{\lambda_k t} v_k = c_k \underbrace{\left(e^{t\Re\lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{it\Im\lambda_k}\right)}_{\text{frequency}} v_k$$

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- The frequency of the vibrations depend on

  $$e^{it\Im\lambda_k} = \cos(t\Im\lambda_k) + i\sin(t\Im\lambda_k),$$

  therefore the imaginary part of λ_k.

Eigenvalues, motivation

Stability

Eigenvalues, motivation

Stability

- The system $y'(t) = Ay(t)$ is called asymptotically stable if for all initial conditions $y(0) \in \mathbb{R}^n$

$$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$

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- Asymptotic stability is equivalent to

$$e^{t\Re\lambda_k} \to 0 \text{ as } t \to \infty \quad \Longleftrightarrow \quad \Re\lambda_k < 0$$

for each $k = 1, \dots, n$

Eigenvalues, motivation

The system $y'(t) = Ay(t)$ is asymptotically stable

$$\Longleftrightarrow$$

All of the eigenvalues of A have negative real parts

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Example:

The system

$$y'(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} y(t)$$

with eigenvalues $\lambda_1 = -2$, $\lambda_2 = -1$ is asymptotically stable.

Eigenvalues, motivation

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- When a periodic external force is applied to the system closely matching the natural frequency of the system, the system can exhibit extreme vibrations (*i.e.* $\|y(t)\|$ is very large) even at small t.
- This phenomenon is known as *resonance*.