Eigenvalues - Basics

Emre Mengi

Department of Mathemtics
Koç University
Istanbul, Turkey

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Definition (Eigenvalues and Eigenvectors)

Let $A \in \mathbb{C}^{n\times n}$. Suppose that

$$Ax = \lambda x$$

for some scalar $\lambda \in \mathbb{C}$ and nonzero vector $x \in \mathbb{C}^n$. Then

(i) λ is called an eigenvalue of A, and

(ii) x is called an eigenvector of A associated with λ.

Example:

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1} = \underbrace{1}_{\lambda_1} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_1} \quad \text{and} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2} = \underbrace{3}_{\lambda_2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{x_2}$$

$\lambda_1 = 1$ and $\lambda_2 = 3$ are eigenvalues of A.

$x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors assoc with λ_1, λ_2.

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Eigenvalues and Polynomial Root Finding

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

$$\lambda \textit{ is an eigenvalue of } A \iff \det(A - \lambda I) = 0$$

Proof:

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff Ax = \lambda x \;\; \exists x \neq 0 \\ &\iff Ax - \lambda x = (A - \lambda I)x = 0 \;\; \exists x \neq 0 \\ &\iff A - \lambda I \;\text{ is singular} \\ &\iff \det(A - \lambda I) = 0 \end{aligned}$$

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Example:

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I)) &= \det\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1-\lambda & 4 \\ 1 & -1-\lambda \end{bmatrix}\right) \\ &= (-1-\lambda)^2 - 4 = \lambda^2 + 2\lambda - 3 \end{aligned}$$

Eigenvalues of A

$\det(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1)$,

so the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 1$.

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Definition (Characteristic Polynomial)

$p(\lambda) = \det(A - \lambda I)$ is a monic polynomial of λ of degree n and called the *characteristic polynomial* of A.

e.g.

The characteristic polynomial for $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$

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The eigenvalues of $A \in \mathbb{C}^{n \times n}$ are the roots of its characteristic polynomial.

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Eigenvalues and Polynomial Root Finding

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

- Consider any polynomial of degree n
  $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \quad$ where $a_n \neq 0$.

- Define the monic polynomial $\tilde{p}(z) = p(z)/a_n$.

$$\begin{aligned}\tilde{p}(z) &= z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \cdots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} \\ &= z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0\end{aligned}$$

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Eigenvalues and Polynomial Root Finding

Theorem (Roots and Companion Matrices)

λ is a root of $\tilde{p}(z) = z^n + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0$

$$\Longleftrightarrow$$

λ is an eigenvalue of the $n \times n$ companion matrix

$$\mathcal{C} = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}$$

Eigenvalues and Polynomial Root Finding

Proof:
Suppose $\tilde{p}(\lambda) = 0$. Then

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} -b_{n-1}\lambda^{n-1} - b_{n-2}\lambda^{n-2} - \dots - b_0 \\ \lambda^{n-1} \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}.$$

Consequently, λ is an eigenvalue of $\mathcal{C}$.

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=
$$

$$
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Consequently, λ is an eigenvalue of C.

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for some $v \neq 0$. Then

- $v_{k+1} = \lambda v_k \implies v_{k+1} = \lambda^k v_1, \quad k = 1, \dots, n-1$
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$$\begin{aligned} & -b_{n-1}v_n - b_{n-2}v_{n-1} \cdots - b_1 v_2 - b_0 v_1 = \lambda v_n \\ \implies & -(\lambda^{n-1} b_{n-1} + \lambda^{n-2} b_{n-2} + \cdots + \lambda b_1 + b_0) v_1 = \lambda^n v_1 \\ \implies & \tilde{p}(\lambda) v_1 = 0 \end{aligned}$$

implying λ is a root of $\tilde{p}(z)$.

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- $$\begin{aligned} & -b_{n-1} v_n - b_{n-2} v_{n-1} \cdots - b_1 v_2 - b_0 v_1 = \lambda v_n \\ \implies & -(\lambda^{n-1} b_{n-1} + \lambda^{n-2} b_{n-2} + \cdots + \lambda b_1 + b_0) v_1 = \lambda^n v_1 \\ \implies & \tilde{p}(\lambda) v_1 = 0 \end{aligned}$$

implying λ is a root of $\tilde{p}(z)$.

Eigenvalues and Polynomial Root Finding

Example:

Consider $p(z) = z^2 + 2z - 3$ with the roots $\lambda_1 = -3, \lambda_2 = 1$.

The associated companion matrix is

$$\mathcal{C} = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}$$

with the characteristic polynomial

$$\det(\mathcal{C} - \lambda I) = \det \begin{pmatrix} -2-\lambda & 3 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 3$$

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- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
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Algebraic Multiplicity

Theorem (Eigenvalues and Characteristic Polynomial)

$$\lambda \textit{ is an eigenvalue of } A \iff \det(A - \lambda I) = 0$$

Corollary of the Theorem

Since

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$$

is a polynomial of degree n, A has n (possibly complex) eigenvalues (counting the multiplicities).

Definition (Algebraic Multiplicity)

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$. The multiplicity of λ as a root of $p(\lambda) = \det(A - \lambda I)$ is called the algebraic multip. of λ.

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Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n \times n}$.
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Find an eigenvector v_1 associated with $\lambda_1 = -3$ (below $c \neq 0$)

$$\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$

$$\implies \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

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Eigenspace

Definition (Eigenspace)

Let λ be an eigenvalue of $A \in \mathbf{C}^{n \times n}$. The set $E_\lambda = \mathrm{Null}(A - \lambda I)$ is called the eigenspace of A associated with λ.

- E_λ = (set of eigenvectors of A assoc. with λ) $\cup \{0\}$
- E_λ is also called an *invariant subspace* of A, since

$$x \in E_\lambda \implies Ax = \lambda x \in E_\lambda$$

that is $\quad \{Ax : x \in E_\lambda\} \subseteq E_\lambda$.

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e.g.

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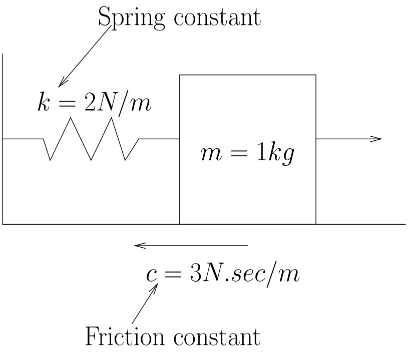
Mass-Spring Systems



Motion of vibrating structures is governed by eigenvalues.

- By Newton's law of motion

$$\text{Net Force} = ma(t)$$

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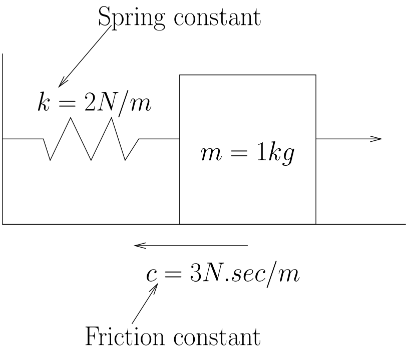

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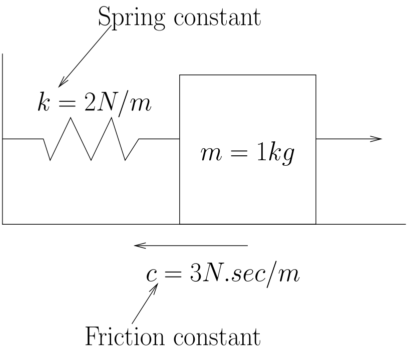

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Notation

$$x(t) : \text{displacement} \qquad v(t) = x'(t) : \text{velocity}$$
$$a(t) = x''(t) : \text{acceleration}$$

Combining the equations for the net force yields

$$ma(t) = -cv(t) - kx(t)$$
$$\Longrightarrow$$
$$mx''(t) = -cx'(t) - kx(t)$$
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$$\underbrace{\begin{bmatrix} v'(t) \\ x'(t) \end{bmatrix}}_{y'(t)} = \underbrace{\begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} v(t) \\ x(t) \end{bmatrix}}_{y(t)}$$

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

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- The solution for the system $y'(t) = Ay(t)$ is of the form

$$\begin{aligned} y(t) &= c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \\ &= c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{aligned}$$

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Differential Equations

- Suppose $A \in \mathbb{R}^{n \times n}$. Consider the differential equation

$$y'(t) = Ay(t).$$

- Assume that A has n distinct eigenvalues.
  - Denote the eigenvalues with $\lambda_1, \dots, \lambda_n$, and
  - the associated eigenvectors with $v_1, \dots, v_n$.

- The solution $y(t) : \mathbb{R} \to \mathbb{C}^n$ is of the form

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$$c_k e^{\lambda_k t} v_k = c_k \underbrace{\left(e^{t\Re\lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{it\Im\lambda_k}\right)}_{\text{frequency}} v_k$$

- The amplitude of the vibrations (*i.e.* $\|y(t)\|$) depend on $e^{t\Re\lambda_k}$, therefore the real part of λ_k.
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- The system $y'(t) = Ay(t)$ is called asymptotically stable if for all initial conditions $y(0) \in \mathbb{R}^n$
$$y(t) \to 0 \quad \text{as} \quad t \to \infty.$$
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$\Longleftrightarrow$

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Example:
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