# **Eigenvalues - Basics**

#### Emre Mengi

Department of Mathemtics Koç University Istanbul, Turkey

December 5th, 2011

ヘロト 人間 とくほとくほとう

E 990

Emre Mengi

#### Definition (Eigenvalues and Eigenvectors)

Let  $A \in \mathbb{C}^{n \times n}$ . Suppose that

$$Ax = \lambda x$$

for some scalar  $\lambda \in \mathbb{C}$  and nonzero vector  $x \in \mathbb{C}^n$ . Then (i)  $\lambda$  is called an eigenvalue of *A*, and (ii) *x* is called an eigenvector of *A* associated with  $\lambda$ .

イロト イヨト イヨト イ

#### Example:



 $\lambda_1 = 1$  and  $\lambda_2 = 3$  are eigenvalues of *A*.

 $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are eigenvectors assoc with  $\lambda_1$ ,  $\lambda_2$ .

・ロン・(理)・ ・ ヨン・

#### Example:



 $\lambda_1 = 1$  and  $\lambda_2 = 3$  are eigenvalues of *A*.

 $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are eigenvectors assoc with  $\lambda_1, \lambda_2$ .

イロト イポト イヨト イヨト 三日

Emre Mengi

#### Example:



イロト イポト イヨト イヨト 一臣

 $\lambda_1 = 1$  and  $\lambda_2 = 3$  are eigenvalues of *A*.

 $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are eigenvectors assoc with  $\lambda_1$ ,  $\lambda_2$ .

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)  $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

 $\begin{array}{lll} \frac{Proot:}{\lambda \text{ is an eigenvalue of } A} & \iff & Ax = \lambda x \ \exists x \neq 0 \\ \Leftrightarrow & Ax - \lambda x = (A - \lambda I)x = 0 \ \exists x \neq 0 \\ \Leftrightarrow & A - \lambda I \text{ is singular} \\ \Leftrightarrow & \det(A - \lambda I) = 0 \end{array}$ 

・ロト ・聞 ト ・ ヨト ・ ヨト … ヨ

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

 $\begin{array}{ll} \hline 1 & 1 \\ \hline \lambda \text{ is an eigenvalue of } A & \iff & Ax = \lambda x \ \exists x \neq 0 \\ \Leftrightarrow & Ax - \lambda x = (A - \lambda I)x = 0 \ \exists x \neq 0 \\ \Leftrightarrow & A - \lambda I \text{ is singular} \\ \Leftrightarrow & \det(A - \lambda I) = 0 \end{array}$ 

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

#### Proof:

 $\lambda \text{ is an eigenvalue of } A \iff Ax = \lambda x \exists x \neq 0$  $\iff Ax - \lambda x = (A - \lambda I)x = 0 \exists x \neq 0$  $\iff A - \lambda I \text{ is singular}$  $\iff \det(A - \lambda I) = 0$ 

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

#### Proof:

 $\lambda$  is an eigenvalue of  $A \iff Ax = \lambda x \exists x \neq 0$ 

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

Proof:

 $\lambda$  is an eigenvalue of A

 $\begin{array}{ll} \Longleftrightarrow & Ax = \lambda x \ \exists x \neq 0 \\ \Leftrightarrow & Ax - \lambda x = (A - \lambda I)x = 0 \ \exists x \neq 0 \\ \Leftrightarrow & A - \lambda I \text{ is singular} \\ \Leftrightarrow & \det(A - \lambda I) = 0 \end{array}$ 

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

Proof:

 $\lambda$  is an eigenvalue of A

For any eigenvalue problem there is an equivalent polynomial root-finding problem.

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

Proof:

 $\lambda$  is an eigenvalue of A

 $\begin{array}{ll} \Longleftrightarrow & Ax = \lambda x \ \exists x \neq 0 \\ \Leftrightarrow & Ax - \lambda x = (A - \lambda I)x = 0 \ \exists x \neq 0 \\ \Leftrightarrow & A - \lambda I \text{ is singular} \\ \Leftrightarrow & \det(A - \lambda I) = 0 \end{array}$ 

イロト イポト イヨト イヨト 一臣

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det \left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= det \left( \begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix} \right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

Eigenvalues of A det $(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1),$ 

so the eigenvalues are  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

イロト イポト イヨト イヨト

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det\left(\begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$
$$= det\left(\begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix}\right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

Eigenvalues of A det(A -  $\lambda I$ ) =  $\lambda^2$  + 2 $\lambda$  - 3 = ( $\lambda$  + 3)( $\lambda$  - 1),

so the eigenvalues are  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

イロト イポト イヨト イヨト

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det \left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= det \left( \begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix} \right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

<u>Eigenvalues of A</u> det $(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1),$ 

so the eigenvalues are  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

イロト イポト イヨト イヨト

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det \left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= det \left( \begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix} \right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

Eigenvalues of A det $(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1),$ 

so the eigenvalues are  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

イロト イポト イヨト イヨト

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det \left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= det \left( \begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix} \right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

 $\frac{\text{Eigenvalues of } A}{\det(A - \lambda I) = \lambda^2} + 2\lambda - 3 = (\lambda + 3)(\lambda - 1),$ 

so the eigenvalues are  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

・ロン ・聞と ・ ほと ・ ほとう

ъ

#### **Eigenvalues and Polynomial Root Finding**

$$\frac{\text{Example:}}{A = \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix}}$$

$$det(A - \lambda I)) = det \left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$
$$= det \left( \begin{bmatrix} -1 - \lambda & 4 \\ 1 & -1 - \lambda \end{bmatrix} \right)$$
$$= (-1 - \lambda)^2 - 4 = \lambda^2 + 2\lambda - 3$$

Eigenvalues of A  $det(A - \lambda I) = \lambda^2 + 2\lambda - 3 = (\lambda + 3)(\lambda - 1),$ so the eigenvalues are  $\lambda_1 = -3, \lambda_2 = 1.$ 

ヘロト ヘアト ヘビト ヘビト

ъ

# Eigenvalues and Polynomial Root Finding

#### Definition (Characteristic Polynomial)

 $p(\lambda) = \det(A - \lambda I)$  is a monic polynomial of  $\lambda$  of degree *n* and called the *characteristic polynomial* of *A*.

# e.g. The characteristic polynomial for $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + 2\lambda - 3$$

The eigenvalues of  $A \in \mathbb{C}^{n \times n}$  are the roots of its characteristic polynomial.

#### Emre Mengi

#### **Eigenvalues and Polynomial Root Finding**

#### Definition (Characteristic Polynomial)

 $p(\lambda) = \det(A - \lambda I)$  is a monic polynomial of  $\lambda$  of degree *n* and called the *characteristic polynomial* of *A*.

#### e.g.

The characteristic polynomial for  $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$ 

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + 2\lambda - 3$$

The eigenvalues of  $A \in \mathbb{C}^{n \times n}$  are the roots of its characteristic polynomial.

#### Definition (Characteristic Polynomial)

 $p(\lambda) = \det(A - \lambda I)$  is a monic polynomial of  $\lambda$  of degree *n* and called the *characteristic polynomial* of *A*.

#### e.g.

The characteristic polynomial for  $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$ 

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + 2\lambda - 3$$

The eigenvalues of  $A \in \mathbb{C}^{n \times n}$  are the roots of its characteristic polynomial.

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

- Consider any polynomial of degree n $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  where  $a_n \neq 0$ .
- Define the monic polynomial  $\tilde{p}(z) = p(z)/a_n$ .

$$\tilde{p}(Z) = z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} = z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0$$

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

• Consider any polynomial of degree n $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  where  $a_n \neq 0$ .

• Define the monic polynomial  $\tilde{p}(z) = p(z)/a_n$ .

Emre Mengi

$$\tilde{\mathcal{D}}(Z) = Z^{n} + \frac{a_{n-1}}{a_n} Z^{n-1} + \dots + \frac{a_1}{a_n} Z + \frac{a_0}{a_n} \\ = Z^{n} + b_{n-1} Z^{n-1} + \dots + b_1 Z + b_0$$

イロト イポト イヨト イヨト 一日

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

• Consider any polynomial of degree n $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  where  $a_n \neq 0$ .

・ロト ・聞 ト ・ ヨト ・ ヨト … ヨ

- Define the monic polynomial  $\tilde{p}(z) = p(z)/a_n$ .
  - $\tilde{\rho}(z) = z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_1}{a_n} z + \frac{a_0}{a_n}$  $= z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0$

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

• Consider any polynomial of degree n $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  where  $a_n \neq 0$ .

・ロト ・聞 ト ・ ヨト ・ ヨトー

• Define the monic polynomial  $\tilde{p}(z) = p(z)/a_n$ .

$$\tilde{p}(z) = z^{n} + \frac{a_{n-1}}{a_{n}} z^{n-1} + \dots + \frac{a_{1}}{a_{n}} z + \frac{a_{0}}{a_{n}} \\ = z^{n} + b_{n-1} z^{n-1} + \dots + b_{1} z + b_{0}$$

For any polynomial there is an equivalent eigenvalue problem whose eigenvalues are same as the roots of the polynomial.

• Consider any polynomial of degree n $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$  where  $a_n \neq 0$ .

・ロト ・聞 ト ・ ヨト ・ ヨトー

• Define the monic polynomial  $\tilde{p}(z) = p(z)/a_n$ .

$$\tilde{p}(z) = z^n + \frac{a_{n-1}}{a_n} z^{n-1} + \dots + \frac{a_1}{a_n} z + \frac{a_0}{a_n} = z^n + b_{n-1} z^{n-1} + \dots + b_1 z + b_0$$

Theorem (Roots and Companion Matrices)  $\lambda \text{ is a root of } \tilde{p}(z) = z^n + b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots + b_1z + b_0$ 

 $\lambda$  is an eigenvalue of the n imes n companion matrix

]	$-b_{n-1}$	$-b_{n-2}$		$-b_1$	$-b_0$
	1	0		0	0
$\mathcal{C} =  $	0	1		0	0
	÷		۰.		:
	0	0		1	0

<ロ> <問> <問> < E> < E> < E> < E

#### **Eigenvalues and Polynomial Root Finding**

<u>Proof:</u> Suppose  $\tilde{p}(\lambda) = 0$ . Then



ヘロン ヘアン ヘビン ヘビン

#### Eigenvalues and Polynomial Root Finding

Proof: Suppose  $\tilde{p}(\lambda) = 0$ . Then



<ロ> (四) (四) (三) (三) (三) (三)

#### Eigenvalues and Polynomial Root Finding

Proof: Suppose  $\tilde{p}(\lambda) = 0$ . Then



< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

#### Eigenvalues and Polynomial Root Finding

Proof: Suppose  $\tilde{p}(\lambda) = 0$ . Then



< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

# Eigenvalues and Polynomial Root Finding

<u>Proof:</u> Suppose  $\tilde{p}(\lambda) = 0$ . Then

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} -b_{n-1}\lambda^{n-1} - b_{n-2}\lambda^{n-2} - \dots - b_0 \\ \lambda^{n-1} \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

イロト イヨト イヨト イ

프 🕨 🗆 프

# **Eigenvalues and Polynomial Root Finding**

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$v_{k+1} = \lambda v_k \implies v_{k+1} = \lambda^k v_1, \quad k = 1, \dots, n-1$$

$$\begin{array}{l} -b_{n-1}v_n - b_{n-2}v_{n-1}\cdots - b_1v_2 - b_0v_1 = \lambda v_n \\ \Longrightarrow & -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_1 \\ \Longrightarrow & \tilde{\rho}(\lambda)v_1 = 0 \end{array}$$

イロト 不得 とくほ とくほ とう

3

implying  $\lambda$  is a root of  $\tilde{p}(z)$ .

# Eigenvalues and Polynomial Root Finding

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$\mathbf{v}_{k+1} = \lambda \mathbf{v}_k \implies \mathbf{v}_{k+1} = \lambda^k \mathbf{v}_1, \quad k = 1, \dots, n-1$$

$$\begin{array}{l} -b_{n-1}v_n - b_{n-2}v_{n-1}\cdots - b_1v_2 - b_0v_1 = \lambda v_n \\ \Rightarrow \quad -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_1 \\ \Rightarrow \quad \tilde{p}(\lambda)v_1 = 0 \end{array}$$

◆□▶ ◆舂▶ ◆臣▶ ◆臣▶ ─臣

implying  $\lambda$  is a root of  $\tilde{p}(z)$ .

# **Eigenvalues and Polynomial Root Finding**

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$\mathbf{v}_{k+1} = \lambda \mathbf{v}_k \implies \mathbf{v}_{k+1} = \lambda^k \mathbf{v}_1, \quad k = 1, \dots, n-1$$

$$-b_{n-1}v_n - b_{n-2}v_{n-1}\cdots - b_1v_2 - b_0v_1 = \lambda v_n -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_n \tilde{c}(\lambda)v_n = 0$$

<ロ> (四) (四) (三) (三) (三)

implying  $\lambda$  is a root of  $\tilde{p}(z)$ .

# Eigenvalues and Polynomial Root Finding

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$v_{k+1} = \lambda v_k \implies v_{k+1} = \lambda^k v_1, \quad k = 1, \dots, n-1$$

$$\begin{array}{l} -b_{n-1}v_n - b_{n-2}v_{n-1} \cdots - b_1v_2 - b_0v_1 = \lambda v_n \\ \Longrightarrow & -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_1 \\ \Longrightarrow & \tilde{\rho}(\lambda)v_1 = 0 \end{array}$$

イロト イポト イヨト イヨト

3

implying  $\lambda$  is a root of  $\tilde{\rho}(z)$ .
# Eigenvalues and Polynomial Root Finding

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$v_{k+1} = \lambda v_k \implies v_{k+1} = \lambda^k v_1, \quad k = 1, \dots, n-1$$

$$\begin{array}{l} -b_{n-1}v_n - b_{n-2}v_{n-1}\cdots - b_1v_2 - b_0v_1 = \lambda v_n \\ \Longrightarrow & -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_1 \\ \Longrightarrow & \tilde{p}(\lambda)v_1 = 0 \end{array}$$

э

implying  $\lambda$  is a root of  $\tilde{p}(z)$ .

# Eigenvalues and Polynomial Root Finding

Conversely, suppose

$$\begin{bmatrix} -b_{n-1} & -b_{n-2} & \dots & -b_1 & -b_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix} v = \lambda v$$

for some  $v \neq 0$ . Then

• 
$$v_{k+1} = \lambda v_k \implies v_{k+1} = \lambda^k v_1, \quad k = 1, \dots, n-1$$

$$\begin{array}{l} -b_{n-1}v_n - b_{n-2}v_{n-1}\cdots - b_1v_2 - b_0v_1 = \lambda v_n \\ \Longrightarrow & -(\lambda^{n-1}b_{n-1} + \lambda^{n-2}b_{n-2} + \cdots + \lambda b_1 + b_0)v_1 = \lambda^n v_1 \\ \Longrightarrow & \tilde{p}(\lambda)v_1 = 0 \end{array}$$

イロト 不得 とくほと くほとう

3

implying  $\lambda$  is a root of  $\tilde{p}(z)$ .

Example: Consider  $p(z) = z^2 + 2z - 3$  with the roots  $\lambda_1 = -3, \lambda_2 = 1$ .

The associated companion matrix is

$$C = \left[ \begin{array}{rr} -2 & 3 \\ 1 & 0 \end{array} \right]$$

with the characteristic polynomial

$$\det(\mathcal{C} - \lambda I) = \det\begin{pmatrix} -2 - \lambda & 3\\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 3$$

・ロト ・聞 と ・ ヨ と ・ ヨ と 。

Example: Consider  $p(z) = z^2 + 2z - 3$  with the roots  $\lambda_1 = -3, \lambda_2 = 1$ .

### The associated companion matrix is

$$\mathcal{C} = \left[ \begin{array}{rr} -2 & 3 \\ 1 & 0 \end{array} \right]$$

with the characteristic polynomial

$$\det(\mathcal{C} - \lambda I) = \det\begin{pmatrix} -2 - \lambda & 3\\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 3$$

イロト イポト イヨト イヨト

Example: Consider  $p(z) = z^2 + 2z - 3$  with the roots  $\lambda_1 = -3, \lambda_2 = 1$ .

The associated companion matrix is

$$\mathcal{C} = \left[ egin{array}{cc} -2 & 3 \ 1 & 0 \end{array} 
ight]$$

with the characteristic polynomial

$$\det(\mathcal{C} - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 3\\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 3$$

・ロン・西方・ ・ ヨン・ ヨン・

Example: Consider  $p(z) = z^2 + 2z - 3$  with the roots  $\lambda_1 = -3, \lambda_2 = 1$ .

The associated companion matrix is

$$\mathcal{C} = \left[ egin{array}{cc} -2 & 3 \ 1 & 0 \end{array} 
ight]$$

with the characteristic polynomial

$$\det(\mathcal{C} - \lambda I) = \det \begin{pmatrix} -2 - \lambda & 3\\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 3$$

・ロン・西方・ ・ ヨン・ ヨン・

- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).

イロト イポト イヨト イヨ

- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).

ヘロト ヘアト ヘヨト ヘ

- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).

ヘロト ヘアト ヘヨト ヘ

- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).

イロト イポト イヨト イヨト

- It was shown by N.H. Abel (in the 19th century) that there is no algebraic formula for the roots of a polynomial of degree > 4.
- Consequently, there can be no algorithm that can compute eigenvalues exactly in finitely many iterations.
  - If there was such an algorithm, then the roots of any polynomial could be computed by means of the companion matrix.
  - This would imply the existence of an algebraic formula for the roots of a polynomial (Contradicts with N. H. Abel's result).

イロト イポト イヨト イヨト

# Algebraic Multiplicity

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

### Corollary of the Theorem

Since

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + \dots + a_1 \lambda + a_0$$

is a polynomial of degree *n*, *A* has *n* (possibly complex) eigenvalues (counting the multiplicities).

### Definition (Algebraic Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The multiplicity of  $\lambda$  as a root of  $p(\lambda) = \det(A - \lambda I)$  is called the algebraic multip. of  $\lambda$ .

# Algebraic Multiplicity

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

### Corollary of the Theorem

Since

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + \cdots + a_1 \lambda + a_0$$

is a polynomial of degree n, A has n (possibly complex) eigenvalues (counting the multiplicities).

### Definition (Algebraic Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The multiplicity of  $\lambda$  as a root of  $p(\lambda) = \det(A - \lambda I)$  is called the algebraic multip. of  $\lambda$ .

# Algebraic Multiplicity

Theorem (Eigenvalues and Characteristic Polynomial)

 $\lambda$  is an eigenvalue of  $A \iff \det(A - \lambda I) = 0$ 

### Corollary of the Theorem

Since

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + \dots + a_1 \lambda + a_0$$

is a polynomial of degree n, A has n (possibly complex) eigenvalues (counting the multiplicities).

### Definition (Algebraic Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The multiplicity of  $\lambda$  as a root of  $p(\lambda) = \det(A - \lambda I)$  is called the algebraic multip. of  $\lambda$ .

# Calculation of Eigenvectors

### Calculation of Eigenvectors

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . Then v is an eigenvector associated with  $\lambda \iff (A - \lambda I)v = 0$  and  $v \neq 0$ .

Example:  
The matrix 
$$A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$$
 has eigenvalues  $\lambda_1 = -3$ ,  $\lambda_2 = 1$ .

・ロン・西方・ ・ ヨン・ ヨン・

# Calculation of Eigenvectors

### Calculation of Eigenvectors

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . Then v is an eigenvector associated with  $\lambda \iff (A - \lambda I)v = 0$ and  $v \neq 0$ .

# Example: The matrix $A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = -3, \lambda_2 = 1$ .

# Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\left( \begin{bmatrix} -1 & 4\\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} 2 & 4\\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\implies \boxed{v_1 = c \begin{bmatrix} -2\\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

### Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\implies \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

### Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\Longrightarrow \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

# Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\Longrightarrow \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

・ロト ・四ト ・ヨト ・ヨト

$$\left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

## Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\Longrightarrow \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

イロト イポト イヨト イヨト

$$\left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

## Calculation of Eigenvectors

Find an eigenvector  $v_1$  associated with  $\lambda_1 = -3$  (below  $c \neq 0$ )

$$\begin{pmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{pmatrix} v_1 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} v_1 = 0$$
$$\Longrightarrow \boxed{v_1 = c \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

Finding an eigenvector  $v_2$  associated with  $\lambda_2 = 1$  (below  $c \neq 0$ )

・ロト ・四ト ・ヨト ・ヨト

$$\left( \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v_1 = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} v_2 = 0$$
$$\implies \boxed{v_2 = c \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

Let  $\lambda$  be an eigenvalue of  $A \in \mathbf{C}^{n \times n}$ . The set  $E_{\lambda} = \text{Null}(A - \lambda I)$  is called the eigenspace of A associated with  $\lambda$ .

### • $E_{\lambda}$ = (set of eigenvectors of *A* assoc. with $\lambda$ ) $\cup$ {0}

E<sub>λ</sub> is also called an *invariant subspace* of A, since
 x ∈ E<sub>λ</sub> ⇒ Ax = λx ∈ E<sub>λ</sub>
 that is {Ax : x ∈ E<sub>λ</sub>} ⊆ E<sub>λ</sub>.

・ロト ・聞 と ・ ヨ と ・ ヨ と …



Let  $\lambda$  be an eigenvalue of  $A \in \mathbf{C}^{n \times n}$ . The set  $E_{\lambda} = \text{Null}(A - \lambda I)$  is called the eigenspace of A associated with  $\lambda$ .

### *E*<sub>λ</sub> = (set of eigenvectors of *A* assoc. with λ) ∪ {0}

*E<sub>λ</sub>* is also called an *invariant subspace* of *A*, since
 *x* ∈ *E<sub>λ</sub>* ⇒ *Ax* = λ*x* ∈ *E<sub>λ</sub>* that is {*Ax* : *x* ∈ *E<sub>λ</sub>*} ⊆ *E<sub>λ</sub>*.

・ロト ・聞 ト ・ ヨト ・ ヨト … ヨ



Let  $\lambda$  be an eigenvalue of  $A \in \mathbf{C}^{n \times n}$ . The set  $E_{\lambda} = \text{Null}(A - \lambda I)$  is called the eigenspace of A associated with  $\lambda$ .

- $E_{\lambda}$  = (set of eigenvectors of *A* assoc. with  $\lambda$ )  $\cup$  {0}
- *E<sub>λ</sub>* is also called an *invariant subspace* of *A*, since
   *x* ∈ *E<sub>λ</sub>* ⇒ *Ax* = λ*x* ∈ *E<sub>λ</sub>* that is {*Ax* : *x* ∈ *E<sub>λ</sub>*} ⊆ *E<sub>λ</sub>*.

◆□▶ ◆□▶ ◆三▶ ◆三▶ ● ● ●

Let  $\lambda$  be an eigenvalue of  $A \in \mathbf{C}^{n \times n}$ . The set  $E_{\lambda} = \text{Null}(A - \lambda I)$  is called the eigenspace of A associated with  $\lambda$ .

- *E*<sub>λ</sub> = (set of eigenvectors of *A* assoc. with λ) ∪ {0}
- *E<sub>λ</sub>* is also called an *invariant subspace* of *A*, since
   *x* ∈ *E<sub>λ</sub> ⇒ Ax* = λx ∈ *E<sub>λ</sub>* that is {*Ax* : *x* ∈ *E<sub>λ</sub>*} ⊆ *E<sub>λ</sub>*.

Let  $\lambda$  be an eigenvalue of  $A \in \mathbf{C}^{n \times n}$ . The set  $E_{\lambda} = \text{Null}(A - \lambda I)$  is called the eigenspace of A associated with  $\lambda$ .

- $E_{\lambda}$  = (set of eigenvectors of *A* assoc. with  $\lambda$ )  $\cup$  {0}
- $E_{\lambda}$  is also called an *invariant subspace* of A, since  $x \in E_{\lambda} \implies Ax = \lambda x \in E_{\lambda}$ that is  $\{Ax : x \in E_{\lambda}\} \subseteq E_{\lambda}.$

・ロト ・聞 ト ・ ヨト ・ ヨト … ヨ

# **Geometric Multiplicity**

*e.g.*  
For 
$$A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$$
 with the eigenvalues  $\lambda_1 = -3, \lambda_2 = 1$   
 $E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$  and  $E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$ 

### Definition (Geometric Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The dimension of the eigenspace  $E_{\lambda} = \text{Null}(A - \lambda I)$  associated with  $\lambda$  is called the geometric multiplicity of  $\lambda$ .

イロン イボン イヨン イヨン

# **Geometric Multiplicity**

e.g.  
For 
$$A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$$
 with the eigenvalues  $\lambda_1 = -3$ ,  $\lambda_2 = 1$   
 $E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$  and  $E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ .

### Definition (Geometric Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The dimension of the eigenspace  $E_{\lambda} = \text{Null}(A - \lambda I)$  associated with  $\lambda$  is called the geometric multiplicity of  $\lambda$ .

イロン イボン イヨン イヨン

# **Geometric Multiplicity**

e.g.  
For 
$$A = \begin{bmatrix} -1 & 4 \\ 1 & -1 \end{bmatrix}$$
 with the eigenvalues  $\lambda_1 = -3$ ,  $\lambda_2 = 1$   
 $E_{\lambda_1} = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$  and  $E_{\lambda_2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ .

### Definition (Geometric Multiplicity)

Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $A \in \mathbb{C}^{n \times n}$ . The dimension of the eigenspace  $E_{\lambda} = \text{Null}(A - \lambda I)$  associated with  $\lambda$  is called the geometric multiplicity of  $\lambda$ .

ヘロト ヘアト ヘビト ヘビト

ъ

# Mass-Spring Systems



Motion of vibrating structures is governed by eigenvalues.

$$c = 3N.sec/m$$
  
Friction constant

- By Newton's law of motion
   Net Force —
- The friction and springs apply forces againstadisplacement

# Mass-Spring Systems



Motion of vibrating structures is governed by eigenvalues.

$$c = 3N.sec/m$$
  
Friction constant

By Newton's law of motion

Net Force = ma(t)

• The friction and springs apply forces against-displacement

# Mass-Spring Systems



Motion of vibrating structures is governed by eigenvalues.

$$c = 3N.sec/m$$
  
Friction constant

By Newton's law of motion

Net Force = ma(t)

The friction and springs apply forces against displacement

# Mass Spring Systems

### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

# Combining the equations for the net force yields ma(t) = -cv(t) - kx(t) $\implies$ mx''(t) = -cx'(t) - kx(t) $\implies$ x''(t) = -3x'(t) - 2x(t) $\implies$ x''(t) + 3x'(t) + 2x(t) = 0

# Mass Spring Systems

### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

 $a_{1}(t)$ 

# Combining the equations for the net force yields

mo(t)

$$ma(t) = -cv(t) - kx(t)$$

$$\implies$$

$$mx''(t) = -cx'(t) - kx(t)$$

$$\implies$$

$$x''(t) = -3x'(t) - 2x(t)$$

$$\implies$$

$$x''(t) + 3x'(t) + 2x(t) = 0$$

 $L_{1}(+)$ 

3

# Mass Spring Systems

### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

### Combining the equations for the net force yields

$$ma(t) = -cv(t) - kx(t)$$

$$\implies$$

$$mx''(t) = -cx'(t) - kx(t)$$

$$\implies$$

$$x''(t) = -3x'(t) - 2x(t)$$

$$\implies$$

$$x''(t) + 3x'(t) + 2x(t) = 0$$

2
Basic Definitions Motivation

## Mass Spring Systems

#### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

#### Combining the equations for the net force yields

$$ma(t) = -cv(t) - kx(t)$$

$$\implies$$

$$mx''(t) = -cx'(t) - kx(t)$$

$$\implies$$

$$x''(t) = -3x'(t) - 2x(t)$$

$$\implies$$

$$x''(t) + 3x'(t) + 2x(t) = 0$$

3

Emre Mengi

Basic Definitions Motivation

## Mass Spring Systems

#### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

#### Combining the equations for the net force yields

$$ma(t) = -cv(t) - kx(t)$$

$$\implies$$

$$mx''(t) = -cx'(t) - kx(t)$$

$$\implies$$

$$x''(t) = -3x'(t) - 2x(t)$$

$$\implies$$

$$x''(t) + 3x'(t) + 2x(t) = 0$$

### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

### The equation of motion

$$x''(t) + 3x'(t) + 2x(t) = 0$$

can be expressed in terms of v(t) and x(t).

$$v'(t) + 3v(t) + 2x(t) = 0$$
  
 $-v(t) + x'(t) = 0$ 

イロン 不得 とくほ とくほとう

### Notation

$$x(t)$$
: displacement  $v(t) = x'(t)$ : velocity  
 $a(t) = x''(t)$ : acceleration

### The equation of motion

$$x''(t) + 3x'(t) + 2x(t) = 0$$

can be expressed in terms of v(t) and x(t).

$$v'(t) + 3v(t) + 2x(t) = 0$$
  
 $-v(t) + x'(t) = 0$ 

イロン 不得 とくほ とくほとう

Basic Definitions Motivation

### Mass Spring Systems

$$\underbrace{\left[\begin{array}{c} \mathbf{v}'(t)\\ \mathbf{x}'(t) \end{array}\right]}_{\mathbf{y}'(t)} = \underbrace{\left[\begin{array}{c} -3 & -2\\ 1 & 0 \end{array}\right]}_{\mathbf{A}} \underbrace{\left[\begin{array}{c} \mathbf{v}(t)\\ \mathbf{x}(t) \end{array}\right]}_{\mathbf{y}(t)}$$

$$A = \left[ \begin{array}{rr} -3 & -2 \\ 1 & 0 \end{array} \right]$$

has the eigenvalues  $\lambda_1 = -2$  and  $\lambda_2 = -1$ with the assoc. eigenvectors  $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

ヘロン 人間 とくほど くほとう

Basic Definitions Motivation

## Mass Spring Systems

$$\underbrace{\left[\begin{array}{c} v'(t) \\ x'(t) \end{array}\right]}_{y'(t)} = \underbrace{\left[\begin{array}{c} -3 & -2 \\ 1 & 0 \end{array}\right]}_{A} \underbrace{\left[\begin{array}{c} v(t) \\ x(t) \end{array}\right]}_{y(t)}$$

$$A = \left[ \begin{array}{rr} -3 & -2 \\ 1 & 0 \end{array} \right]$$

has the eigenvalues  $\lambda_1 = -2$  and  $\lambda_2 = -1$ with the assoc. eigenvectors  $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$  and  $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

◆□ > ◆□ > ◆豆 > ◆豆 > -

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{array}{lll} \begin{split} \mathbf{y}'(t) &=& \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &=& A \mathbf{v}_1 (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (\mathbf{c}_2 \mathbf{e}^{\lambda_2 t}) \\ &=& A (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &=& A \mathbf{y}(t) \end{split}$$

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{array}{lll} \begin{split} \mathbf{y}'(t) &=& \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &=& A \mathbf{v}_1 (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (\mathbf{c}_2 \mathbf{e}^{\lambda_2 t}) \\ &=& A (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &=& A \mathbf{y}(t) \end{split}$$

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{aligned} \mathbf{v}'(t) &= \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &= A \mathbf{v}_1 (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (\mathbf{c}_2 \mathbf{e}^{\lambda_2 t}) \\ &= A (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &= A \mathbf{y}(t) \end{aligned}$$

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{aligned} \mathbf{y}'(t) &= \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &= A \mathbf{v}_1 (c_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (c_2 \mathbf{e}^{\lambda_2 t}) \\ &= A (c_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + c_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &= A \mathbf{y}(t) \end{aligned}$$

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$y'(t) = \lambda_1 v_1 c_1 e^{\lambda_1 t} + \lambda_2 v_2 c_2 e^{\lambda_2 t}$$
  
=  $Av_1(c_1 e^{\lambda_1 t}) + Av_2(c_2 e^{\lambda_2 t})$   
=  $A(c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2)$   
=  $Ay(t)$ 

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{aligned} \mathbf{v}'(t) &= \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &= A \mathbf{v}_1 (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (\mathbf{c}_2 \mathbf{e}^{\lambda_2 t}) \\ &= A (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &= A \mathbf{y}(t) \end{aligned}$$

• The solution for the system y'(t) = Ay(t) is of the form

Basic Definitions Motivation

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$
  
=  $c_1 e^{-2t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$ 

• Verify that  $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$  is a solution

$$\begin{aligned} \mathbf{v}'(t) &= \lambda_1 \mathbf{v}_1 \mathbf{c}_1 \mathbf{e}^{\lambda_1 t} + \lambda_2 \mathbf{v}_2 \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \\ &= A \mathbf{v}_1 (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t}) + A \mathbf{v}_2 (\mathbf{c}_2 \mathbf{e}^{\lambda_2 t}) \\ &= A (\mathbf{c}_1 \mathbf{e}^{\lambda_1 t} \mathbf{v}_1 + \mathbf{c}_2 \mathbf{e}^{\lambda_2 t} \mathbf{v}_2) \\ &= A \mathbf{y}(t) \end{aligned}$$

• Suppose  $A \in \mathbb{R}^{n \times n}$ . Consider the differential equation

$$y'(t) = Ay(t).$$

• Assume that *A* has *n* distinct eigenvalues.

- Denote the eigenvalues with  $\lambda_1, \ldots, \lambda_n$ , and
- the associated eigenvectors with  $v_1, \ldots, v_n$ .
- The solution  $y(t) : \mathbb{R} \to \mathbb{C}^n$  is of the form

 $y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$ 

ヘロト 人間 とくほとく ほとう

• Suppose  $A \in \mathbb{R}^{n \times n}$ . Consider the differential equation

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t).$$

### • Assume that A has n distinct eigenvalues.

- Denote the eigenvalues with  $\lambda_1, \ldots, \lambda_n$ , and
- the associated eigenvectors with  $v_1, \ldots, v_n$ .
- The solution  $y(t) : \mathbb{R} \to \mathbb{C}^n$  is of the form

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$$

• Suppose  $A \in \mathbb{R}^{n \times n}$ . Consider the differential equation

$$y'(t) = Ay(t).$$

#### • Assume that A has n distinct eigenvalues.

- Denote the eigenvalues with  $\lambda_1, \ldots, \lambda_n$ , and
- the associated eigenvectors with  $v_1, \ldots, v_n$ .
- The solution  $y(t) : \mathbb{R} \to \mathbb{C}^n$  is of the form

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

Consider an eigenvalue  $\lambda_k = \Re \lambda_k + i \Im \lambda_k$  where  $\Re \lambda_k, \Im \lambda_k \in \mathbb{R}$ .

$$C_k e^{\lambda_k t} V_k = C_k \underbrace{\left(e^{t\Re\lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{it\Im\lambda_k}\right)}_{\text{frequency}} V_k$$

The amplitude of the vibrations (*i.e.* ||*y*(*t*)||) depend on e<sup>t ℜλ<sub>k</sub></sup>, therefore the real part of λ<sub>k</sub>.

イロト イヨト イヨト イ

• The frequency of the vibrations depend on  $e^{it\Im\lambda_k} = \cos(t\Im\lambda_k) + i\sin(t\Im\lambda_k),$ therefore the imaginary part of  $\lambda_k$ .

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

Consider an eigenvalue  $\lambda_k = \Re \lambda_k + i \Im \lambda_k$  where  $\Re \lambda_k, \Im \lambda_k \in \mathbb{R}$ .

$$c_k e^{\lambda_k t} v_k = c_k \underbrace{\left(e^{t \Re \lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{i t \Im \lambda_k}\right)}_{\text{frequency}} v_k$$

The amplitude of the vibrations (*i.e.* ||*y*(*t*)||) depend on e<sup>t ℜλk</sup>, therefore the real part of λk.

• • • • • • • • • • • •

• The frequency of the vibrations depend on  $e^{it\Im\lambda_k} = \cos(t\Im\lambda_k) + i\sin(t\Im\lambda_k)$ 

therefore the imaginary part of  $\lambda_k$ 

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

Consider an eigenvalue  $\lambda_k = \Re \lambda_k + i \Im \lambda_k$  where  $\Re \lambda_k, \Im \lambda_k \in \mathbb{R}$ .

$$C_k e^{\lambda_k t} v_k = C_k \underbrace{\left(e^{t \Re \lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{it \Im \lambda_k}\right)}_{\text{frequency}} v_k$$

• The amplitude of the vibrations (*i.e.* ||y(t)||) depend on  $e^{t\Re\lambda_k}$ , therefore the real part of  $\lambda_k$ .

• The frequency of the vibrations depend on  $e^{it\Im\lambda_k} = \cos(t\Im\lambda_k) + i\sin(t\Im\lambda_k),$ therefore the imaginary part of  $\lambda_k.$ 

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \cdots + c_n e^{\lambda_n t} v_n$$

Consider an eigenvalue  $\lambda_k = \Re \lambda_k + i \Im \lambda_k$  where  $\Re \lambda_k, \Im \lambda_k \in \mathbb{R}$ .

$$C_k e^{\lambda_k t} v_k = C_k \underbrace{\left(e^{t \Re \lambda_k}\right)}_{\text{amplitude}} \underbrace{\left(e^{it \Im \lambda_k}\right)}_{\text{frequency}} v_k$$

- The amplitude of the vibrations (*i.e.* ||y(t)||) depend on e<sup>t ℜλk</sup>, therefore the real part of λk.
- The frequency of the vibrations depend on

$$e^{it\Im\lambda_k} = \cos(t\Im\lambda_k) + i\sin(t\Im\lambda_k),$$

イロト イポト イヨト イヨト 三日

therefore the imaginary part of  $\lambda_k$ .



 The system y'(t) = Ay(t) is called asymptotically stable if for all initial conditions y(0) ∈ ℝ<sup>n</sup>

 $y(t) \to 0$  as  $t \to \infty$ .

◆□ > ◆□ > ◆豆 > ◆豆 > -

• Asymptotic stability is equivalent to  $e^{t\Re\lambda_k} \to 0 \text{ as } t \to \infty \iff \Re\lambda_k < 0$ for each k = 1, ..., n



 The system y'(t) = Ay(t) is called asymptotically stable if for all initial conditions y(0) ∈ ℝ<sup>n</sup>

y(t) 
ightarrow 0 as  $t 
ightarrow \infty$ .

・ロト ・聞 ト ・ ヨト ・ ヨト … ヨ

• Asymptotic stability is equivalent to  $e^{t\Re\lambda_k} \to 0 \text{ as } t \to \infty \iff \Re\lambda_k < 0$ for each k = 1, ..., n



 The system y'(t) = Ay(t) is called asymptotically stable if for all initial conditions y(0) ∈ ℝ<sup>n</sup>

$$y(t) 
ightarrow 0$$
 as  $t 
ightarrow \infty$ .

• Asymptotic stability is equivalent to

 $e^{t\Re\lambda_k} o 0$  as  $t o \infty \iff \Re\lambda_k < 0$ 

・ロン・西方・ ・ ヨン・ ヨン・

э.

for each  $k = 1, \ldots, n$ 

## Stability

#### Asymptotic Stability

The system y'(t) = Ay(t) is asymptotically stable  $\iff$ All of the eigenvalues of *A* have negative real parts

Example: The system

$$y'(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} y(t)$$

<ロ> (四) (四) (三) (三) (三)

with eigenvalues  $\lambda_1 = -2$ ,  $\lambda_2 = -1$  is asymptotically stable.

## Stability

#### Asymptotic Stability

The system y'(t) = Ay(t) is asymptotically stable  $\iff$ All of the eigenvalues of *A* have negative real parts

Example: The system

$$y'(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} y(t)$$

イロト 不得 とくほと くほとう

with eigenvalues  $\lambda_1 = -2$ ,  $\lambda_2 = -1$  is asymptotically stable.