LECTURE 22

SIMILARITY TRANSFORMATIONS

A similarity transformation

$$T : A \longrightarrow SAS^{-1}$$

(Above $A \in \mathbb{C}^{n \times n}$ and $S \in \mathbb{C}^{n \times n}$ is invertible)

EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{with} \quad S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

The transformed matrix

$$SAS^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Characteristic polynomials

$$\det(A - \lambda I) = (1-\lambda)^2$$

$$\det(SAS^{-1} - \lambda I) = (1-\lambda)^2$$

Eigenvalues

* A has $\lambda = 1$ as an eigenvalue of algebraic multiplicity two.

* SAS^{-1} has $\lambda = 1$ as an eigenvalue of algebraic multiplicity two.

Geometric multiplicities

Eigenspace of A assoc with $\lambda = 1$.

$$\text{Null}(A - \lambda I) = \text{Null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

Eigenspace of SAS^{-1} assoc with $\lambda = 1$

$$\text{Null}(SAS^{-1} - \lambda I) = \text{Null}\left(\begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

Geometric multiplicity of $\lambda = 1$ as an eigenvalue of A and SAS^{-1} is one.

THM

Let $A, B \in \mathbb{C}^{n \times n}$ be two similar matrices. Then A and B have the same set of eigenvalues with same algebraic and geometric multiplicities

PROOF

There exists an invertible $S \in \mathbb{C}^{n \times n}$ such that

$$B = SAS^{-1}.$$

Comparing the characteristic polynomials of A and B

$$\begin{aligned} \det(B - \lambda I) &= \det(SAS^{-1} - \lambda I) \\ &= \det(SAS^{-1} - \lambda SS^{-1}) \\ &= \det(S)\det(A-\lambda I)\det(S^{-1}) \\ &= \det(A - \lambda I) \end{aligned}$$

Consequently A and B have the same eigenvalues with same algebraic multiplicities.

Moreover

$$\begin{aligned} \operatorname{rank}(B - \lambda I) &= \operatorname{rank}(S(A-\lambda I)S^{-1}) \\ &= \operatorname{rank}(A - \lambda I) \quad \left(\text{SINCE } S \text{ IS FULL RANK}\right) \end{aligned}$$

$$\Longrightarrow$$

$$\dim(\operatorname{Null}(B - \lambda I)) = \dim(\operatorname{Null}(A - \lambda I))$$

Consequently geometric multiplicities of all eigenvalues of A and B are the same. $\square$

Recall the example

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\lambda = 1$ is the only eigenvalue with algebraic mult. 2
geometric mult. 1

algebraic mult > geometric mult.

THM

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $A \in \mathbb{C}^{n\times n}$.
Then

$$\text{algebraic multiplicity of } \lambda \geqslant \text{geometric multiplicity of } \lambda$$

PROOF

Let $B = \{q_1, \dots, q_r\}$ be an orthonormal basis for $\mathrm{Null}(A - \lambda I)$ (i.e. geom. multiplicity of λ is r)

Append vectors to orthonormal basis B to obtain an orthonormal basis for $\mathbb{C}^n$

$$\{q_1, \dots, q_r, \hat{q}_{r+1}, \dots \hat{q}_n\}$$

Then

$$Q = [\underbrace{q_1 \cdots q_r}_{Q_r} \; \underbrace{\hat{q}_{r+1} \cdots \hat{q}_n}_{\hat{Q}_{n-r}}]$$

is unitary.

Consider

$$Q^* A Q = \begin{bmatrix} Q_r^* A Q_r & Q_r^* A \hat{Q}_{n-r} \\ \hat{Q}_{n-r}^* A Q_r & \hat{Q}_{n-r}^* A \hat{Q}_{n-r} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda Q_r^* Q_r & Q_r^* A \hat{Q}_{n-r} \\ \lambda \hat{Q}_{n-r}^* Q_r & \hat{Q}_{n-r}^* A \hat{Q}_{n-r} \end{bmatrix}$$

$$\left(\begin{array}{c} \underline{\text{NOTE}} \\ \hat{q}_j^* q_k = 0 \\ j = r+1, \cdots, n \\ k = 1, \cdots, r \end{array} \right) = \begin{bmatrix} \lambda I_r & Q_r^* A \hat{Q}_{n-r} \\ 0 & \hat{Q}_{n-r}^* A \hat{Q}_{n-r} \end{bmatrix}$$

Noting that A and $Q^* A Q$ are similar

$$\det(A - zI) = \det(Q^* A Q - zI)$$

$$= (\lambda - z)^r \det(\hat{Q}_{n-r}^* A \hat{Q}_{n-r} - zI)$$

Consequently algebraic mult of λ $\geq$ r = geometric mult of λ □

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TERMINOLOGY

Consider

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$\lambda = 1$ is an eigenvalue (DEFECTIVE)
alg mult $= 2 >$ geom mult $= 1$

$\lambda = 2$ is an eigenvalue (SEMI-SIMPLE)
alg mult $= 2 =$ geom mult $= 2$

$\lambda = 3$ is an eigenvalue (SIMPLE)
alg mult $= 1$

An eigenvalue λ is called ~~[illegible]~~

(i) DEFECTIVE if
alg mult of $\lambda >$ geom mult of λ

(ii) SEMI-SIMPLE if
alg mult of $\lambda =$ geom mult of λ
AND alg mult of $\lambda > 1$

(iii) SIMPLE if
alg mult of $\lambda = 1$

DIAGONALIZABILITY

For eigenvalue computation we will apply similarity transformations to reduce the matrix into a form revealing the eigenvalues.

Unfortunately not all matrices can be reduced into diagonal form by similarity transformations.

EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Suppose there exists V such that

$$VAV^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But then

the geometric multiplicities $\lambda = 1$ as an eigenvalue of A and VAV^{-1} must be one. NOT TRUE FOR VAV^{-1}

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

DEFN (Diagonalizability)

A matrix $A \in \mathbb{C}^{n \times n}$ is called diagonalizable if there exist an invertible matrix $V \in \mathbb{C}^{n \times n}$ such that

$$A = V \Lambda V^{-1}$$

where $\Lambda \in \mathbb{C}^{n \times n}$ is diagonal.

THM

A matrix $A \in \mathbb{C}^{n \times n}$ is diagonalizable if and only if all eigenvalues of A are semi-simple.

PROOF

Suppose A is diagonalizable, i.e., for some V (invertible)

$$A = V \Lambda V^{-1}$$

$$\Longrightarrow$$

$$AV = V\Lambda$$

that is

$$A v_j = \lambda_j v_j \qquad j = 1, \dots, n$$

where $\{v_1, \dots, v_n\}$ is linearly independent.

Consider any eigenvalue λ appearing r times in $\{\lambda_1, \dots, \lambda_n\}$. Denote these eigenvalues by $\lambda_{j_1}, \dots, \lambda_{j_r}$. Then the set $\{v_{j_1}, \dots, v_{j_r}\}$ is linearly independent. Consequently alg. mult of λ = geom. mult of $\lambda = r$, i.e. λ is semi-simple.

Now suppose A has semi-simple eigenvalues. Then A has n linearly independent eigenvectors (Exercise). In particular associated with each eigenvalue λ_j there is an eigenvector v_j s.t.

$$\{v_1, \dots, v_n\}$$

is linearly independent.

Consequently

$$A v_j = \lambda_j v_j \qquad j = 1, \dots, n$$

$$\Longrightarrow A[v_1 \dots v_n] = [\lambda_1 v_1 \dots \lambda_n v_n]$$

$$\Longrightarrow A\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$\Longrightarrow A = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^{-1}$$

that is A is diagonalizable. (Note: above $\{\lambda_1, \ldots, \lambda_n\}$ may have repetitions.)

$\square$

SCHUR FACTORIZATION

All matrices are (unitarily) similar to triangular matrices

EXAMPLE

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

(NOT DIAGONALIZABLE
$\lambda = 2$ is eigval
of alg mult 2
geom mult 1
$E_\lambda = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$)

For

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

we have

$$Q^* A Q = \underbrace{\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}}_{T}$$

Schur factorization for a matrix $A \in \mathbb{C}^{n\times n}$ is of the form

$$A = QTQ^*$$

$T \in \mathbb{C}^{n\times n}$: upper triangular

$Q \in \mathbb{C}^{n\times n}$: unitary

<u>THM</u>

Every matrix $A \in \mathbb{C}^{n\times n}$ has a Schur factorization of the form

$$A = QTQ^*$$

where $Q \in \mathbb{C}^{n\times n}$ is unitary, and $T \in \mathbb{C}^{n\times n}$ is upper triangular.

<u>PROOF</u>

The proof is by induction on the size n.

$\boxed{\text{Base case} - n=1}$

Obvious $a = (1).a.(1)$

$\boxed{\text{Inductive case}}$

Let us suppose that all $(n-1)\times(n-1)$ matrices have Schur factorization.

Let λ be an eigenvalue of A and q be a unit eigenvector assoc. with λ i.e.

$$Aq = \lambda q \quad \text{and} \quad \|q\|_2 = 1$$

Form a unitary matrix

$$Q = [\, q \;\; \underbrace{q_2 \;\; \cdots \;\; q_n}_{\hat{Q}} \,] \in \mathbb{C}^{n\times n}$$

Then

$$Q^* A Q = \begin{bmatrix} q^* A q & q^* A \hat{Q} \\ \hat{Q}^* A q & \hat{Q}^* A \hat{Q} \end{bmatrix}$$

$$= \begin{bmatrix} \lambda q^* q & q^* A \hat{Q} \\ \lambda \hat{Q}^* q & \hat{Q}^* A \hat{Q} \end{bmatrix}$$

$$\left(\begin{array}{c} \underline{\text{NOTE}} \\ q_j^* q = 0 \\ j = 2, \dots, n \\ \Longrightarrow \\ \hat{Q}^* q = 0 \end{array} \right) = \begin{bmatrix} \lambda & q^* A \hat{Q} \\ 0 & \hat{Q}^* A \hat{Q} \end{bmatrix}$$

Now by inductive hypothesis there exists a Schur factor. for $\hat{Q}^* A \hat{Q}$, i.e,

$$\hat{Q}^* A \hat{Q} = \tilde{Q} T \tilde{Q}^*$$

for some unitary $\tilde{Q} \in \mathbb{C}^{(n-1)\times(n-1)}$ and upper triangular $T \in \mathbb{C}^{(n-1)\times(n-1)}$. Consequently

$$Q^* A Q = \begin{bmatrix} \lambda & q^* A \hat{Q} \\ 0 & \tilde{Q} T \tilde{Q}^* \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q} \end{bmatrix} \begin{bmatrix} \lambda & q^* A \hat{Q} \tilde{Q} \\ 0 & T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}^* \end{bmatrix}$$

$$\Longrightarrow$$

$$A = \underbrace{Q \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q} \end{bmatrix}}_{\text{UNITARY}} \underbrace{\begin{bmatrix} \lambda & q^* A \hat{Q} \tilde{Q} \\ 0 & T \end{bmatrix}}_{\text{UPPER TRIANGULAR}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}^* \end{bmatrix} Q^*}_{\text{COMPLEX CONJ. TRANS OF UNITARY}}$$

$\square$

General Strategy for Eigenvalue Computation

Apply unitary similarity transfor. to obtain an upper triangular matrix

$$A \longrightarrow Q_1^* A Q_1 \longrightarrow Q_2^* Q_1^* A Q_1 Q_2 := A_2$$
$$\cdots \longrightarrow Q_m^* \cdots Q_1^* A Q_1 Q_2 \cdots Q_m := A_m$$

Purpose

$$\lim_{m \to \infty} A_m = \underset{\substack{\text{UPPER} \\ \text{TRIANGULAR}}}{T}$$

Infinitely many iterations are required.