LECTURE 23

STAGES OF EIGENVALUE COMPUTATION

STAGE 1 (Reduction into Hessenberg Form)

$$A = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} \times & \cdots & \cdots & \times \\ \times & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & & \times & \times \end{bmatrix}}_{\substack{Q^*AQ \\ \text{in Hessenberg form}}}$$

Q is defined in terms of HH reflectors

FINITELY MANY ITERATIONS

STAGE 2 (QR Algorithm)

$$Q^*AQ = \begin{bmatrix} \times & \cdots & \cdots & \times \\ \times & \ddots & & \vdots \\ & \ddots & \ddots & \vdots \\ 0 & & \times & \times \end{bmatrix}$$

$$\longrightarrow$$

$$\underbrace{Q_k^* \cdots Q_1^* Q^* A Q Q_1 \cdots Q_k}_{A_k} \approx \begin{bmatrix} \times & \cdots & \times \\ & \ddots & \vdots \\ 0 & & \times \end{bmatrix}$$

INFINITELY MANY ITERATIONS

(A_k approaches an upper triangular matrix as $k \to \infty$)

WHY IS STAGE 1 IMPORTANT

* Stage 2 will require computations of QR factorization.
* For a Hessenberg matrix a QR factorization can be computed in $O(n^2)$ time.

REDUCTION INTO HESSENBERG FORM

We will proceed column by column from left to right.

STEP 1 (1st column)

$$A = \begin{bmatrix} \overset{1\times 1}{a} & \overset{1\times(n-1)}{c} \\ \underset{(n-1)\times 1}{b} & \underset{(n-1)\times(n-1)}{D} \end{bmatrix}$$

Let $\hat{Q}_1 \in \mathbb{C}^{(n-1)\times(n-1)}$ be the HH reflector such that

$$b \longrightarrow \begin{bmatrix} \|b\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{Q}_1 b$$

Define

$$Q_1 := \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \in \mathbb{C}^{n \times n} \qquad \left(\frac{\text{Note}}{Q_1^* = Q_1} \right)$$

so that

$$\begin{aligned}
Q_1 A Q_1^* &= Q_1 A Q_1 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \begin{bmatrix} a & c \\ b & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1^* \end{bmatrix} \\
&= \begin{bmatrix} a & c\hat{Q}_1^* \\ \hat{Q}_1 b & \hat{Q}_1 D \hat{Q}_1^* \end{bmatrix} \\
&= \begin{bmatrix} \underbrace{\boxed{X}}_{a} & \overbrace{\boxed{X \; \cdots \; X}}^{c\hat{Q}_1} \\ \underbrace{\begin{matrix} X \\ 0 \\ \vdots \\ 0 \end{matrix}}_{\hat{Q}_1 b} & \underbrace{\begin{matrix} X & \cdots & X \\ X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix}}_{\hat{Q}_1 D \hat{Q}_1} \end{bmatrix}
\end{aligned}$$

STEP k (kth column)

$$\overbrace{Q_{k-1} \cdots\cdots Q_1 A Q_1 \cdots\cdots Q_{k-1}}^{A^{(k)}}$$

$$= \begin{bmatrix} \underbrace{\begin{matrix} X & \cdots & X \\ X & \ddots & \vdots \\ 0 & X & X \end{matrix}}_{\substack{a^{(k)} \\ (k \times k)}} & \underbrace{\begin{matrix} X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix}}_{\substack{c^{(k)} \\ (k \times (n-k))}} \\ \underbrace{\begin{matrix} 0 & \cdots & 0 & X \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & X \end{matrix}}_{\substack{b^{(k)} \\ ((n-k) \times k)}} & \underbrace{\begin{matrix} X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix}}_{\substack{D^{(k)} \\ ((n-k) \times (n-k))}} \end{bmatrix}$$

Let $\widehat{Q}_k \in \mathbb{C}^{(n-k)\times(n-k)}$ be the HH reflector such that

$$b_k^{(k)} \longrightarrow \begin{bmatrix} \|b_k^{(k)}\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \widehat{Q}_k\, b_k^{(k)}$$

Define

$$\overline{Q}_k := \begin{bmatrix} I_k & 0 \\ 0 & \widehat{Q}_k \end{bmatrix} \in \mathbb{C}^{(n-k)\times(n-k)} \qquad \left(\frac{\text{Note}}{Q_k^* = Q_k} \right)$$

so that

$$\begin{aligned}
Q_k A^{(k)} Q_k^* &= Q_k A^{(k)} Q_k \\
&= \begin{bmatrix} I_k & 0 \\ 0 & \widehat{Q}_k \end{bmatrix} \begin{bmatrix} a^{(k)} & c^{(k)} \\ b^{(k)} & D^{(k)} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & \widehat{Q}_k \end{bmatrix} \\
&= \begin{bmatrix} a^{(k)} & c^{(k)}\, \widehat{Q}_k \\ \widehat{Q}_k\, b^{(k)} & \widehat{Q}_k\, D^{(k)}\, \widehat{Q}_k \end{bmatrix}
\end{aligned}$$

$$= \begin{array}{c} k\times k \\ \\ (n-k)\times k \end{array} \left[\begin{array}{c|c} \begin{matrix} X & \cdots & X \\ X & \ddots & \vdots \\ 0 & \ddots X & X \end{matrix} & \begin{matrix} X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix} \\ \hline \begin{matrix} 0 & \cdots & 0 & X \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{matrix} & \begin{matrix} X & \cdots & X \\ X & \cdots & X \\ \vdots & & \vdots \\ X & \cdots & X \end{matrix} \end{array} \right] \begin{array}{c} k\times(n-k) \\ \\ (n-k)\times(n-k) \end{array}$$

ALGORITHM (Reduction into Hessenberg Form)

* Given $A \in \mathbb{C}^{n\times n}$
* Produce a Hessenberg matrix $H \in \mathbb{C}^{n\times n}$ unitarily similar to A

for $k = 1, \ldots, n-2$ (k: col #)

$\quad v = a_{k+1:n,k}$

$\quad u_k = v - \|v\| e_1, \qquad u_k = u_k / \|u_k\|$

$\quad$ (Multiply with Q_k from left; rows $k+1:n$ are affected)

$$a_{k+1:n,\,k:n} = a_{k+1:n,\,k:n} - 2u_k\left(u_k^* \, a_{k+1:n,\,k:n}\right)$$

$\quad$ (Multiply with Q_k from right; cols $k+1:n$ are affected)

$$a_{1:n,\,k+1:n} = a_{1:n,\,k+1:n} - 2\left(a_{1:n,\,k+1:n}\, u_k\right) u_k^*$$

end

$H = A$

COMPUTATIONAL COMPLEXITY

TOTAL # FLOPS $= 10n^3/3$

POWER ITERATION

An algorithm converging to the dominant eigenvector (i.e. eigenvector associated with largest eigenvalue in modulus)

Given $A \in \mathbb{C}^{n \times n}$ and $q_0 \in \mathbb{C}^n$. (where $\|q_0\| = 1$)
This generates a sequence $\{q_k\}$ such that

$$q_k = \frac{A q_{k-1}}{\|A q_{k-1}\|} \qquad k = 1, 2, \ldots$$

Algorithm (Power Iteration)

* Given $A \in \mathbb{C}^{n \times n}$ and $q_0 \in \mathbb{C}^n$ s.t. $\|q_0\| = 1$
* Produce an approximate pair (v, λ) where λ is the eigenvalue with largest modulus and $v \in \mathbb{C}^n$ is the corresponding eigenvector

for $k = 1, \ldots, m$
  $q_k = A q_{k-1}$
  $q_k = q_k / \|q_k\|$
end
$v = q_m, \quad \lambda = q_m^* A q_m$

Convergence of Power Iteration

Since $q_k \in \mathbb{C}^n$ is a unit vector

$$q_k = \frac{A^k q_0}{\|A^k q_0\|}$$

For simplicity assume A is non-defective with

* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ s.t.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

* and associated eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{C}^n$

Since $\{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{C}^n$, we have

$$q_0 = c_1 v_1 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_n \in \mathbb{C}$ meaning

$$q_k = \frac{A^k (c_1 v_1 + \dots + c_n v_n)}{\|A^k (c_1 v_1 + \dots + c_n v_n)\|}$$

Notice that for $j=1,\dots,n$

$$A^2 v_j = A(Av_j) = A(\lambda v_j) = \lambda^2 v_j$$

$$A^3 v_j = A(A^2 v_j) = A(\lambda^2 v_j) = \lambda^3 v_j$$

or more generally

$$A^k v_j = \lambda^k v_j .$$

Consequently

$$q_k = \frac{c_1 \lambda_1^k v_1 + \dots\dots + c_n \lambda_n^k v_n}{\| c_1 \lambda_1^k v_1 + \dots\dots + c_n \lambda_n^k v_n \|}$$

$$= \frac{\lambda_1^k}{|\lambda_1^k|} \frac{c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n}{\left\| c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n \right\|}$$

The terms $\left(\frac{\lambda_2}{\lambda_1}\right)^k, \dots\dots, \left(\frac{\lambda_n}{\lambda_1}\right)^k$ vanish as $k \to \infty$. Therefore

$$\operatorname{span}\{q_k\} \longrightarrow \operatorname{span}\{v_1\} \text{ as } k \to \infty .$$