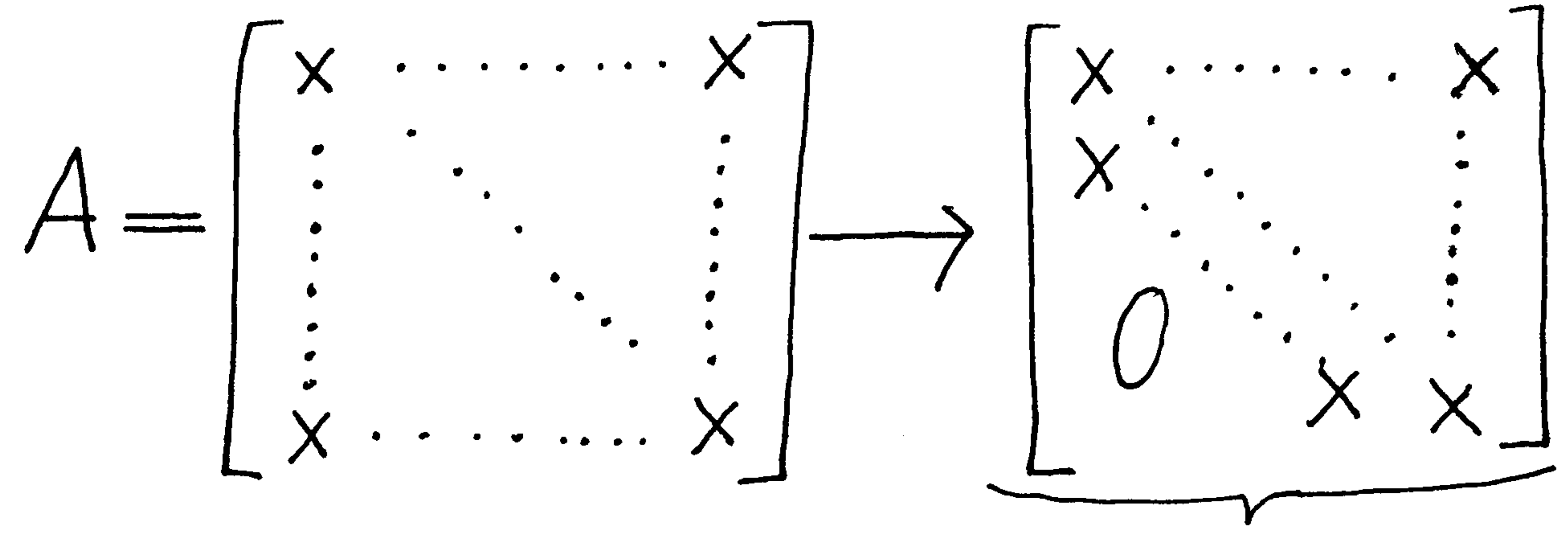


LECTURE 23

STAGES OF EIGENVALUE COMPUTATION

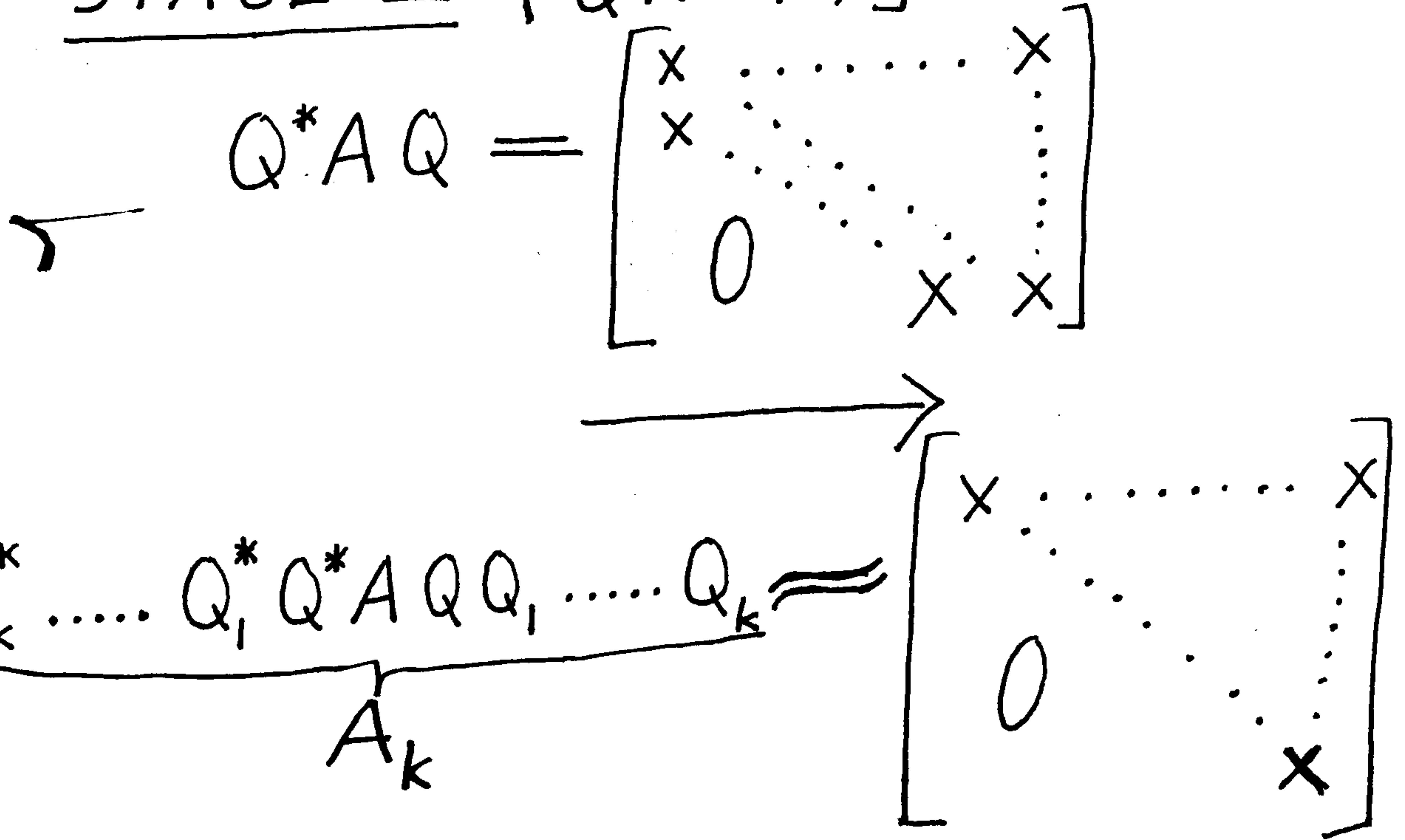
STAGE 1 (Reduction into Hessenberg Form)



Q is defined in terms of HH reflectors $Q^* A Q$ in Hessenberg form

FINITELY MANY ITERATIONS

STAGE 2 (QR Algorithm)



INFINITELY MANY ITERATIONS
 (A_k approaches an upper triangular matrix as $k \rightarrow \infty$) ①

WHY IS STAGE 1 IMPORTANT

- * Stage 2 will require computations of QR factorization.
- * For a Hessenberg matrix a QR factorization can be computed in $O(n^2)$ time.

REDUCTION INTO HESSENBERG FORM

We will proceed column by column from left to right.

STEP 1 (1st column)

$$A = \begin{bmatrix} a & c \\ b & D \end{bmatrix}$$

a is 1×1 , b is $(n-1) \times 1$, c is $1 \times (n-1)$, and D is $(n-1) \times (n-1)$.

Let $\hat{Q}_1 \in \mathbb{C}^{(n-1) \times (n-1)}$ be the HH reflector such that

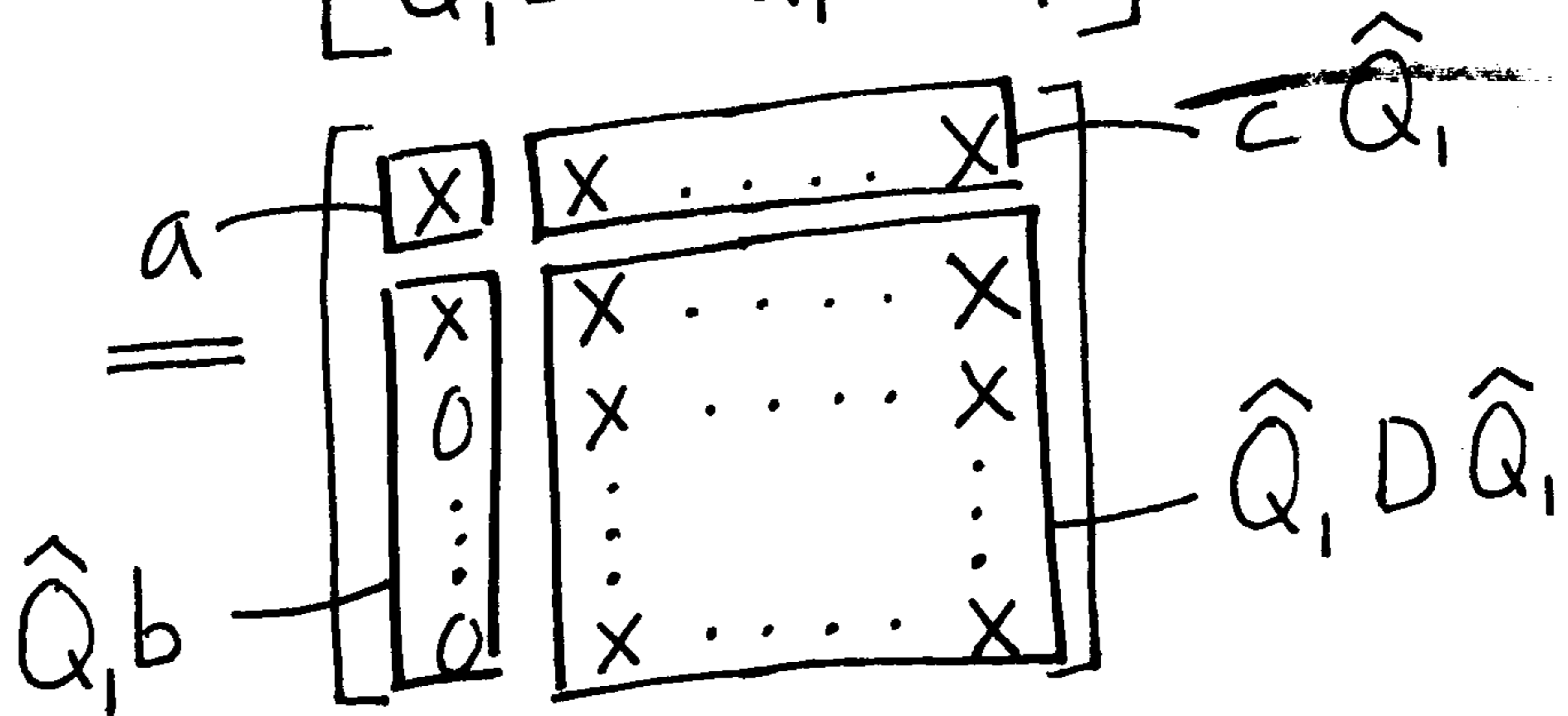
$$b \longrightarrow \begin{bmatrix} \|b\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{Q}_1 b$$

Define

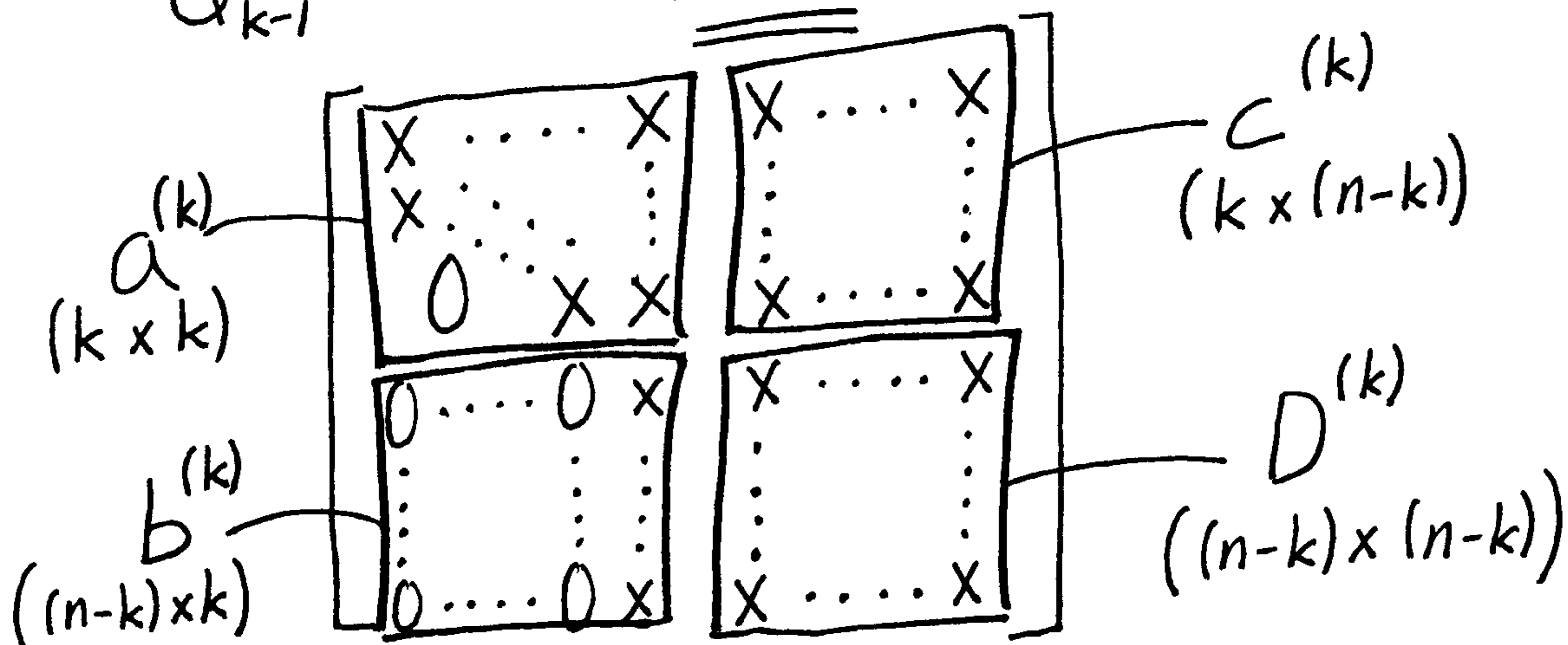
$$Q_1 := \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \in \mathbb{C}^{n \times n} \quad \left(\begin{array}{l} \text{Note} \\ Q_1^* = Q_1 \end{array} \right)$$

so that

$$\begin{aligned} Q_1 A Q_1^* &= Q_1 A Q_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \begin{bmatrix} a & c \\ b & D \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \hat{Q}_1 \end{bmatrix} \\ &= \begin{bmatrix} a & c \hat{Q}_1 \\ \hat{Q}_1 b & \hat{Q}_1 D \hat{Q}_1 \end{bmatrix} \end{aligned}$$



STEP k (kth column) $A^{(k)}$
 $Q_{k-1} \dots Q_1 A Q_1 \dots Q_{k-1}$



Let $\hat{Q}_k \in \mathbb{C}^{(n-k) \times (n-k)}$ be the HH reflector such that

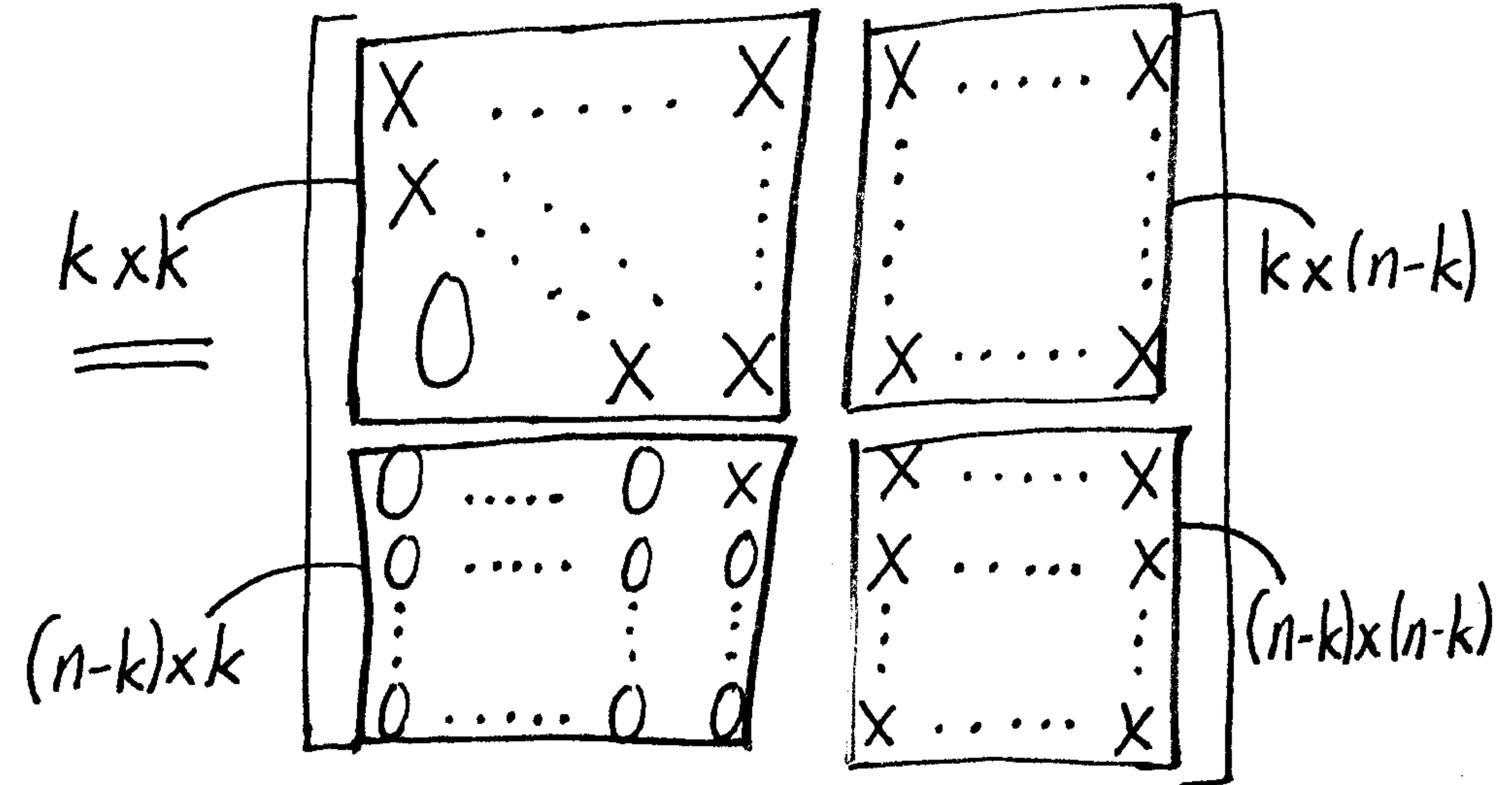
$$b_k^{(k)} \longrightarrow \begin{bmatrix} \|b_k^{(k)}\| \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \hat{Q}_k b_k^{(k)}$$

Define

$$\bar{Q}_k := \begin{bmatrix} I_k & 0 \\ 0 & \hat{Q}_k \end{bmatrix} \in \mathbb{C}^{(n-k) \times (n-k)} \quad \left(\begin{array}{l} \text{Note} \\ Q_k^* = Q_k \end{array} \right)$$

so that

$$\begin{aligned} Q_k A^{(k)} Q_k^* &= Q_k A^{(k)} Q_k \\ &= \begin{bmatrix} I_k & 0 \\ 0 & \hat{Q}_k \end{bmatrix} \begin{bmatrix} a^{(k)} & c^{(k)} \\ b^{(k)} & D^{(k)} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & \hat{Q}_k \end{bmatrix} \\ &= \begin{bmatrix} a^{(k)} & c^{(k)} \hat{Q}_k \\ \hat{Q}_k b^{(k)} & \hat{Q}_k D^{(k)} \hat{Q}_k \end{bmatrix} \end{aligned}$$



ALGORITHM (Reduction into Hessenberg Form)

* Given $A \in \mathbb{C}^{n \times n}$

* Produce a Hessenberg matrix

$H \in \mathbb{C}^{n \times n}$ unitarily similar to A

for $k = 1, \dots, n-2$ (k : col #)

$$v = a_{k+1:n, k}$$

$$u_k = v - \|v\|e_1, \quad u_k = u_k / \|u_k\|$$

(Multiply with Q_k from left;
rows $k+1:n$ are affected)

$$a_{k+1:n, k:n} = a_{k+1:n, k:n} - 2u_k (u_k^* a_{k+1:n, k:n})$$

(Multiply with Q_k from right;
cols $k+1:n$ are affected)

$$a_{1:n, k+1:n} = a_{1:n, k+1:n} - 2(a_{1:n, k+1:n} u_k) u_k^*$$

end

$$H = A$$

COMPUTATIONAL COMPLEXITY

$$\text{TOTAL \# FLOPS} = 10n^3/3$$

POWER ITERATION

An algorithm converging to the dominant eigenvector (eigenvector associated with largest eigenvalue in modulus)

Given $A \in \mathbb{C}^{n \times n}$ and $q_0 \in \mathbb{C}^n$. (where $\|q_0\|=1$)

This generates a sequence $\{q_k\}$ such that

$$q_k = \frac{A q_{k-1}}{\|A q_{k-1}\|} \quad k=1, 2, \dots$$

Algorithm (Power Iteration)

* Given $A \in \mathbb{C}^{n \times n}$ and $q_0 \in \mathbb{C}^n$ s.t. $\|q_0\|=1$

* Produce an approximate pair (v, λ) where λ is the eigenvalue with largest modulus and $v \in \mathbb{C}^n$ is the corresponding eigenvector

for $k=1, \dots, m$

$$q_k = A q_{k-1}$$

$$\text{end } q_k = q_k / \|q_k\|$$

$$v = q_m, \quad \lambda = q_m^* A q_m$$

Convergence of Power Iteration

Since $q_k \in \mathbb{C}^n$ is a unit vector

$$q_k = \frac{A^k q_0}{\|A^k q_0\|}$$

For simplicity assume A is non-defective with

* eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ s.t.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

* and associated eigenvectors $v_1, v_2, \dots, v_n \in \mathbb{C}^n$

Since $\{v_1, v_2, \dots, v_n\}$ is a basis for \mathbb{C}^n , we have

$$q_0 = c_1 v_1 + \dots + c_n v_n$$

for some $c_1, c_2, \dots, c_n \in \mathbb{C}$ meaning

$$q_k = \frac{A^k (c_1 v_1 + \dots + c_n v_n)}{\|A^k (c_1 v_1 + \dots + c_n v_n)\|}$$

Notice that for $j=1, \dots, n$

$$A^2 v_j = A(A v_j) = A(\lambda v_j) = \lambda^2 v_j$$

$$A^3 v_j = A(A^2 v_j) = A(\lambda^2 v_j) = \lambda^3 v_j$$

or more generally

$$A^k v_j = \lambda^k v_j.$$

Consequently

$$\begin{aligned} q_k &= \frac{c_1 \lambda_1^k v_1 + \dots + c_n \lambda_n^k v_n}{\|c_1 \lambda_1^k v_1 + \dots + c_n \lambda_n^k v_n\|} \\ &= \frac{\lambda_1^k}{|\lambda_1^k|} \frac{c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n}{\|c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + \dots + c_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v_n\|} \end{aligned}$$

The terms $\left(\frac{\lambda_2}{\lambda_1}\right)^k, \dots, \left(\frac{\lambda_n}{\lambda_1}\right)^k$ vanish as $k \rightarrow \infty$. Therefore

$\text{span}\{q_k\} \rightarrow \text{span}\{v_1\}$ as $k \rightarrow \infty$.