LECTURE 26

NORMALIZED SIMULTANEOUS ITERATION

When simultaneous iteration is applied all columns of A^k approach the dominant eigenvector of A.
(i.e. A^k is ill-conditioned for large k)

Simultaneous iteration is numerically unstable.

A stable version would normalize at every iteration.

ALGORITHM (Normalized Simultaneous Iteration)

* Given $A \in \mathbb{C}^{n \times n}$ in Hessenberg form
* Produce $Q_\ell \in \mathbb{C}^{n \times n}$ (an estimate for the orthonormal eigenvectors) and diagonal $\Lambda \in \mathbb{C}^{n \times n}$ (with eigenvalues estimates along the diagonal)

Let Q_1 be such that $A = Q_1 R_1$
for $k = 1, \ldots, \ell$
    $Z_k = A Q_k$
    Q_{k+1} is such that $Z_k = Q_{k+1} R_{k+1}$
end
$\Lambda = Q_\ell^* A Q_\ell$

REMARKS

(i) Suppose A is non-singular. Then

$\boxed{\text{CAN BE SHOWN BY INDUCTION}}$ $\quad \mathrm{Range}(A^k) = \mathrm{Range}(Q_k)$

$\Longrightarrow$

The normalized simultaneous iteration is equivalent to the original simultaneous iteration

(ii) Suppose $Q \in \mathbb{C}^{n \times n}$ is the matrix of orthonormal eigenvectors, that is

$$Q = [v_1 \; v_2 \; \cdots \; v_n],$$

then

$$\Lambda = Q^* A Q$$

is diagonal with

$$\Lambda_{ij} = \begin{cases} v_i^* A v_j = \lambda_j v_i^* v_j = 0 & i \neq j \\ v_i^* A v_i = \lambda_i v_i^* v_i = \lambda_i & i = j \end{cases}$$

DERIVATION OF THE QR ALGORITHM FROM THE NORMALIZED SIMULTANEOUS ITERATION

An iteration of the normalized simultaneous iteration

$$(1) \qquad Q_{k+1} R_{k+1} = A Q_k \; (= Z_k)$$

$$(2) \qquad A_{k+1} = Q_{k+1}^* A Q_{k+1}$$

Multiply both sides of eqn (1) by Q_k^* from left

$$\underbrace{Q_k^* Q_{k+1} R_{k+1}}_{\text{QR Factor. of } A_k} = Q_k^* A Q_k = A_k.$$

- We have the QR factorization

$$A_k = \hat{Q}_{k+1} R_{k+1}$$

where

$$\hat{Q}_{k+1} = Q_k^* Q_{k+1}$$

· Relate A_k and A_{k+1}

$$A_k = Q_k^* A Q_k \quad \text{and} \quad A_{k+1} = Q_{k+1}^* A Q_{k+1}$$

$$\boxed{\left(A = Q_k A_k Q_k^*\right)}$$

$$\Longrightarrow$$

$$A_{k+1} = \underbrace{Q_{k+1}^* Q_k}_{\hat{Q}_{k+1}^*} A_k \underbrace{Q_k^* Q_{k+1}}_{\hat{Q}_{k+1}}$$

This can be expressed as an iteration of the QR algorithm

(3) $A_k = \hat{Q}_{k+1} R_{k+1}$

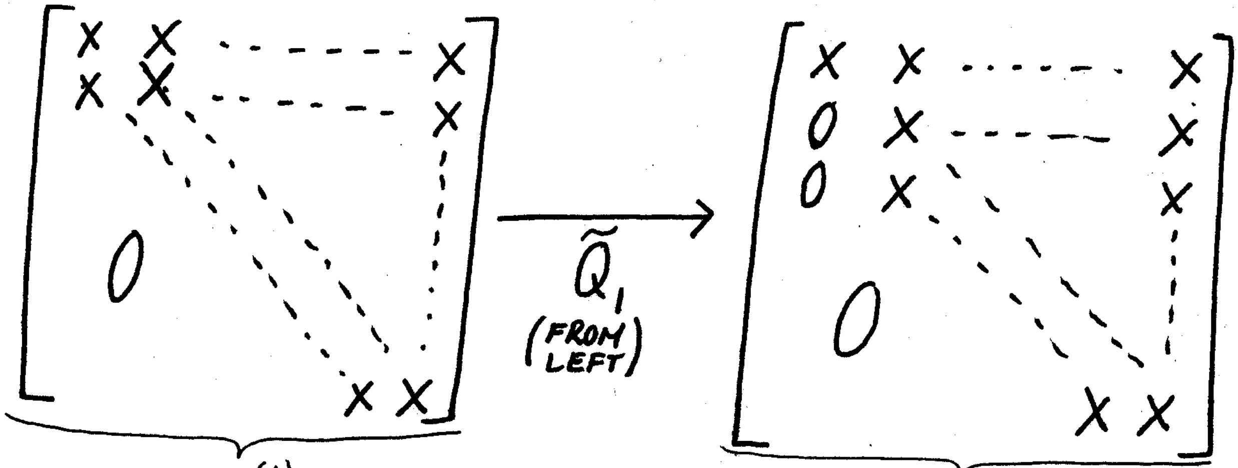
(4) $A_{k+1} = R_{k+1} \hat{Q}_{k+1} \left(= \hat{Q}_{k+1}^* A_k Q_{k+1}\right)$

THE IMPLICIT QR ALGORITHM

Suppose $A_k \in \mathbb{C}^{n \times n}$ is in Hessenberg form.

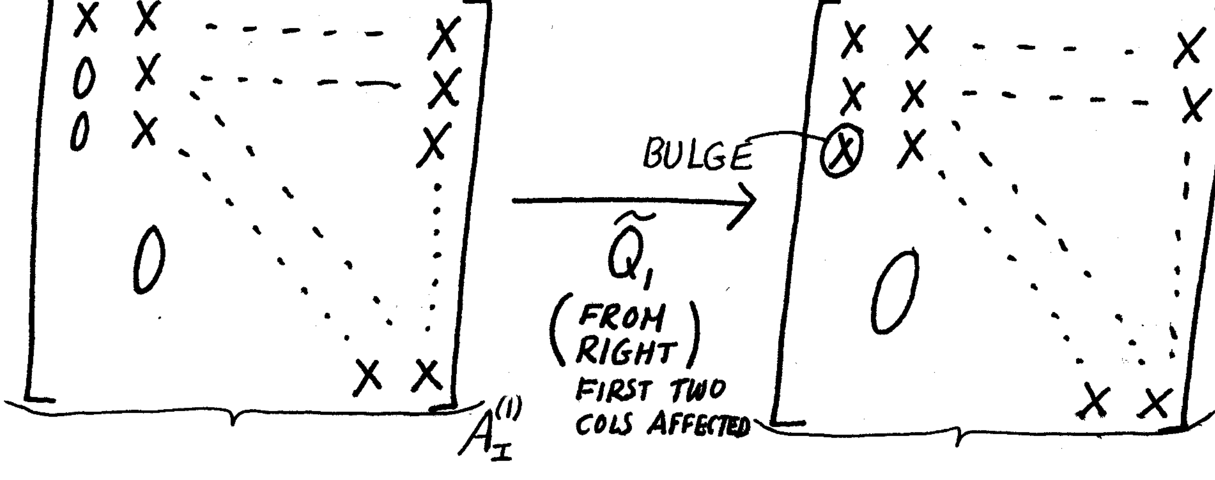
The implicit QR algorithm, at every iteration, proceeds column by column from left to right.

STEP 1 (First Column)



$A_k = A^{(1)}$ $\qquad$ $A_I^{(1)} = \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & I_{n-2} \end{bmatrix} A^{(1)}$

$\tilde{Q}_1$: Householder reflector associated with $A^{(1)}(1:2,1)$; only first two rows are affected



$A_I^{(1)}$ $\qquad$ $A^{(2)} = A_I^{(1)} \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & I_{n-2} \end{bmatrix}$

STEP 2 (Second Column)

BULGE

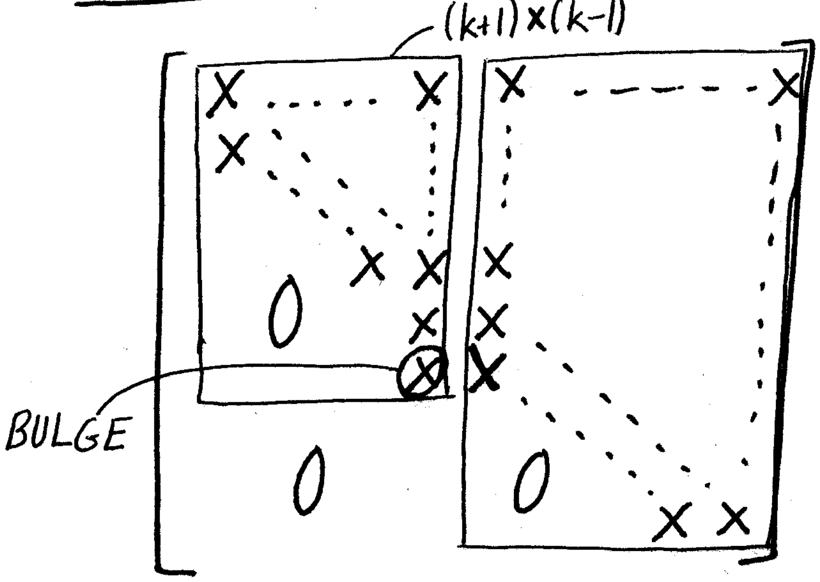
$$\underbrace{\begin{bmatrix} \times & \times & \cdots & \cdots & \times \\ \times & \times & \cdots & \cdots & \times \\ \times & \times & \ddots & & \times \\ 0 & & & \ddots & \vdots \\ \vdots & & 0 & & \vdots \\ 0 & & & \times & \times \end{bmatrix}}_{A^{(2)}} \xrightarrow[\text{(FROM LEFT)}]{\tilde{Q}_2} \underbrace{\begin{bmatrix} \times & \times & \cdots & \cdots & \times \\ \times & \times & \cdots & \cdots & \times \\ 0 & \times & \ddots & & \times \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & \times & \times \end{bmatrix}}_{A_I^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{Q}_2 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix} A^{(2)}}$$

$\tilde{Q}_2$: Householder reflector associated with $A^{(2)}(2:3,1)$; only 2nd and 3rd rows are affected

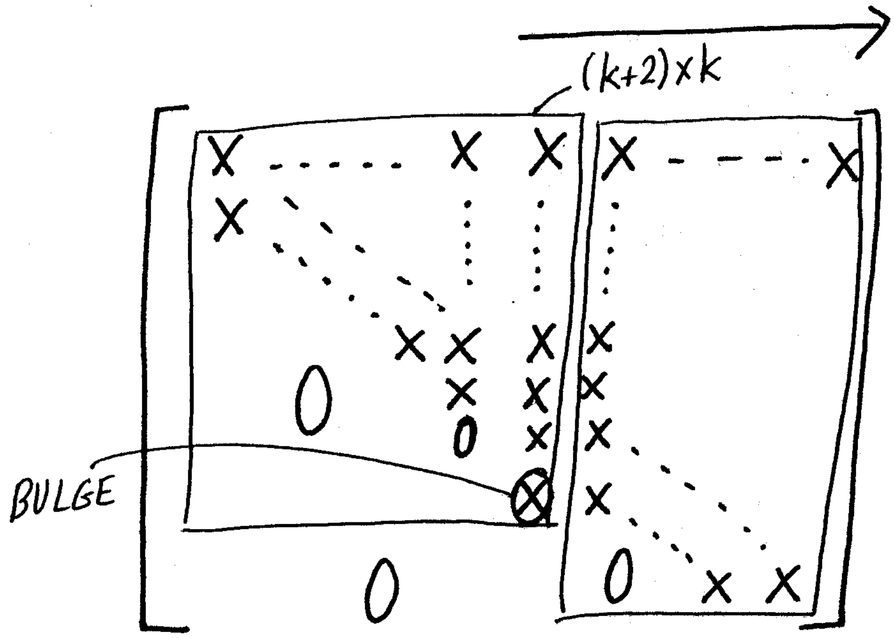
BULGE

$$\underbrace{\begin{bmatrix} \times & \times & \cdots & \cdots & \times \\ \times & \times & \cdots & \cdots & \times \\ 0 & \times & \ddots & & \times \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & \times & \times \end{bmatrix}}_{A_I^{(2)}} \xrightarrow[\substack{\text{(FROM RIGHT;} \\ \text{2nd and 3rd} \\ \text{COLS AFFECTED)}}]{\tilde{Q}_2} \underbrace{\begin{bmatrix} \times & \times & \times & \cdots & \times \\ \times & \times & \times & \cdots & \times \\ 0 & \times & \times & \ddots & \times \\ 0 & \times & \times & \ddots & \times \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & \times & \times \end{bmatrix}}_{A^{(3)} = A_I^{(2)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \tilde{Q}_2 & 0 \\ 0 & 0 & I_{n-3} \end{bmatrix}}$$

STEP k (Bulge on the (k-1)st column)



$$:= A^{(k)}$$



$$= \underbrace{\begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \tilde{Q}_k & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix} A^{(k)} \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \tilde{Q}_k & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}}_{A^{(k+1)}}$$

$\tilde{Q}_k$: Householder reflector associated with $A^{(k)}(k:k+1, k-1)$

* Rows k and (k+1)st are affected when multiplying from left

* Cols k and (k+1)st are affected when multiplying from right
* Bulge moves from (k-1)st col to kth col.

At step $(n-2)$ bulge disappears. $A^{(n-1)}$ is a Hessenberg matrix satisfying

$$A_{k+1} = A^{(n-1)} = \hat{Q}_{n-2} \cdots \hat{Q}_1 A_k \hat{Q}_1 \cdots \hat{Q}_{n-2}$$

where

$$\hat{Q}_k = \begin{bmatrix} I_{k-1} & 0 & 0 \\ 0 & \tilde{Q}_k & 0 \\ 0 & 0 & I_{n-k-1} \end{bmatrix}$$

REMARKS

(i) It can be shown that A_k has a QR factorization of the form

$$A_k = \underbrace{\left(\hat{Q}_1 \hat{Q}_2 \cdots \hat{Q}_{n-2}\right)}_{Q_{k+1}:\text{unitary factor}} R_{k+1}$$

Therefore

$$A_{k+1} = \underbrace{Q^*_{k+1} A_k}_{R_{k+1}} Q_{k+1}$$

equivalently

$$A_{k+1} = R_{k+1} Q_{k+1} \quad \text{and} \quad A_k = Q_{k+1} R_{k+1}$$

- In other words the implicit QR algorithm is equivalent to the explicit QR algorithm.

(ii) In practice the implicit QR algorithm is used together with the shifts. Therefore the HH reflectors are applied to $A_k - \mu_k I$ instead of A_k, i.e.

$$A_{k+1} = \left(\hat{Q}_{n-2} \cdots \hat{Q}_1 (A_k - \mu_k I) \hat{Q}_1 \cdots \hat{Q}_{n-2}\right) + \mu_k I$$