# LECTURE 26

## WHY QR ALGORITHM WORKS

| Power Iteration | Simultaneous Power Iteration |
|---|---|
| $z_k = A^k q_0$ | $Z_k = A^k Q_0$ |
| $q_k = z_k / \lVert z_k \rVert$ | $Q_k$ is s.t. $Z_k = Q_k R_k$ |
| $\hat{\lambda}_k = q_k^* A q_k$ | $\hat{\Lambda}_k = Q_k^* A Q_k$ |

Assumption : $A \in \mathbb{C}^{n \times n}$ is Hermitian

### CONVERGENCE

| Power Iteration | Simultaneous Power Iteration |
|---|---|
| ① $q_k \to v_1$ as $k \to \infty$ | $Q_k \to V = [v_1 \; \cdots \; v_n]$ as $k \to \infty$ |
| ② $\hat{\lambda}_k \to \lambda_1$ as $k \to \infty$ | $\hat{\Lambda}_k \to \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ as $k \to \infty$ |

TO BE DISCUSSED

CLAIM

Simultaneous Power Iteration is equivalent to the QR algorithm, i.e.,

$$\underbrace{A_k}_{\substack{\text{kth estimate} \\ \text{by QR algorithm}}} = \underbrace{\hat{\Lambda}_k}_{\substack{\text{kth estimate} \\ \text{by simultaneous iteration}}}$$

## WHY SIMULTANEOUS ITERATION WORKS

$\lambda_1, \ldots, \lambda_n$ eigenvalues in descending order w.r.t. modulus, i.e.

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$$

Associated orthonormal set of eigenvectors

$$v_1, \ldots, v_n$$

Let

* $S$ be an $m$-dimensional subspace of $\mathbb{R}^n$
* $AS = \{ Ax \mid x \in \mathbb{R}^n \}$

**THM** (Subspace Iteration)

Let $U_m = \text{span}\{v_{m+1}, \dots, v_n\}$, and $S_k := A^k S$. If $S \cap U_m = \{0\}$ and $|\lambda_m| > |\lambda_{m+1}|$

$$S_k \longrightarrow \text{span}\{v_1, \dots, v_m\} \text{ as } k \to \infty.$$

## CONVERGENCE OF SIMULTANEOUS ITERATION

$$S^{(m)} := \text{span}\{e_1, \dots, e_m\} \subseteq \mathbb{C}^n$$

$$S_k^{(m)} := A^k S^{(m)} = \{A^k x \mid x \in S^{(m)}\}$$

By thm-subspace iteration

$$S_k^{(m)} \longrightarrow \text{span}\{v_1, \dots, v_m\} \quad \text{as } k \to \infty$$

under the assumptions

(i) $A$ has distinct eigenvalues

(ii) $S^{(m)} \cap \text{span}\{v_{m+1}, \dots, v_n\} = \{0\}$

But $S_k^{(m)} = \text{Range}(A^k(:, 1{:}m))$ meaning

(*) $\text{Range}(A^k(:, 1{:}m)) \longrightarrow \text{span}\{v_1, \dots, v_m\}$ as $k \to \infty$

# QR Factorization

$$A^{(k)}(:,1:m) = \overbrace{Q}^{n\times m}\,\overbrace{R}^{m\times m}$$

$$= \left[q_1^{(k)} \cdots q_m^{(k)}\right] R$$

<u>NOTE</u>

$\{q_1^{(k)}, \dots, q_m^{(k)}\}$ is an orthonormal basis for Range $(A^{(k)}(:,1:m))$.

Specific instances of $(*)$

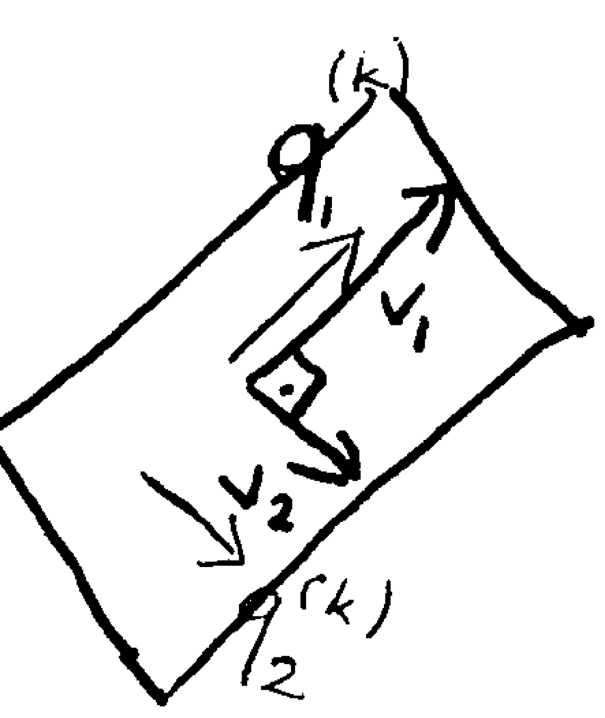
(i) $\text{Range}\,(A^k(:,1)) \longrightarrow \text{span}\{v_1\}$

$\{q_1^{(k)}\}$ is an orthonormal basis for $\text{Range}\,(A^k(:,1))$

$$\Longrightarrow$$

$$q_1^{(k)} \longrightarrow \tilde{c}_{k,1} v_1 \quad \text{as} \quad k\to\infty$$

(ii) $\text{Range}\,(A^k(:,1:2)) \longrightarrow \text{span}\{v_1, v_2\}$




$\{q_1^{(k)}, q_2^{(k)}\}$ is an orthonormal basis for $\text{Range}\,(A^k(:,1:2))$

$\Longrightarrow$ $\left(q_1^{(k)} = \tilde{c}_{k,1} v_1 \text{ and } q_2^{(k)} \perp q_1^{(k)}\right)$

$$q_2^{(k)} \longrightarrow \tilde{c}_{k,2} v_2 \quad \text{as} \quad k\to\infty$$

(iii) $\mathrm{Range}(A^k(:,1:\ell)) \longrightarrow \mathrm{span}\{v_1, \dots, v_\ell\}$

$\{q_1^{(k)}, \dots, q_\ell^{(k)}\}$ is an orthonormal basis for $\mathrm{Range}(A^k(:,1:\ell))$

$$\Longrightarrow \left( \underset{j=1,\dots,\ell-1}{q_j^{(k)} = \tilde{c}_{k,j} v_j} \quad \text{and} \quad q_\ell^{(k)} \perp q_j^{(k)} \right)$$

$$q_\ell^{(k)} \longrightarrow \tilde{c}_{k,\ell} v_\ell \quad \text{as } k \to \infty$$

<u>THM</u> (Simultaneous Iteration)

Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with distinct eigenvalues and

$$A^k = Q_k R_k$$

be a QR factorization. Then generically

$$\mathrm{span}\{Q_k(:,\ell)\} \longrightarrow \mathrm{span}\{v_\ell\} \text{ as } k \to \infty,$$

for $\ell = 1, \dots, n$.

<u>REMARK</u>

For $V = [v_1 \dots v_n]$ we have

$$V^* A V = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$$

i.e.

$$\Lambda_{ij} = \begin{cases} v_i^* A v_j = \lambda_j v_i^* v_j = 0 & i \neq j \\ v_i^* A v_i = \lambda_i v_i^* v_i = \lambda_i & i = j \end{cases}$$

Consequently

$$Q_k^* A Q_k \longrightarrow \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad \text{as } k \to \infty$$

## NORMALIZED SIMULTANEOUS ITERATION

A stable version of simultaneous iteration.

### Algorithm

Let $Q_1$ be s.t. $A = Q_1 R_1$

for $k = 1, \dots, l$

$\quad Z_k = A Q_k$

$\quad Q_{k+1}$ is such that $Z_k = Q_{k+1} R_{k+1}$

end

$\Lambda = Q_{l+1}^* A Q_{l+1}$

### Basic Result

$$\text{Range}(A^k) = \text{Range}(Q_k) \quad \left(\text{For normalized simultaneous iteration}\right)$$

that is

normalized simultaneous iteration is equivalent to simultaneous iteration.

**PROOF**

By induction on $k$ (Exercise).

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simultaneous iteration

$\approx$

normalized simultaneous iteration

$\approx$

QR algorithm

## NORMALIZED SIMULTANEOUS ITERATION AND QR ALGORITHM

An iteration of normalized simultaneous iteration

(a1) $\quad Q_{k+1} R_{k+1} = A Q_k$

(a2) $\quad \hat{\Lambda}_{k+1} = Q_{k+1}^* A Q_{k+1}$

From (a2) and (a1)

$$\hat{\Lambda}_k = Q_k^* A Q_k = \underbrace{Q_k^* Q_{k+1}}_{\hat{Q}_{k+1}} R_{k+1}$$

Now relate $\hat{\Lambda}_k$ and $\hat{\Lambda}_{k+1}$.

$$\begin{aligned} \hat{\Lambda}_{k+1} &= Q_{k+1}^* A Q_{k+1} \\ &= \underbrace{Q_{k+1}^* Q_k}_{\hat{Q}_{k+1}^*} \hat{\Lambda}_k \underbrace{Q_k^* Q_{k+1}}_{\hat{Q}_{k+1}} \end{aligned}$$

Noting $R_{k+1} = \hat{Q}_{k+1}^{*} \hat{\Lambda}_k$ we have

$$\text{(b1)} \quad \hat{\Lambda}_k = \hat{Q}_{k+1} R_{k+1}$$

$$\text{(b2)} \quad \hat{\Lambda}_{k+1} = R_{k+1} \hat{Q}_{k+1}$$