LECTURE 2

MATH 504

FALL 2011

REVIEW ON VECTOR SPACES (Continued)

Linear Independence:

$$\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\} = \left\{c_1\begin{bmatrix}1\\2\end{bmatrix} + c_2\begin{bmatrix}2\\4\end{bmatrix} : c_1, c_2 \in \mathbb{R}\right\}$$

$$= \left\{(c_1+2c_2)\begin{bmatrix}1\\2\end{bmatrix} : c_1, c_2 \in \mathbb{R}\right\}$$

* $\text{span}\underbrace{\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}}_{\text{REDUNDANT}} = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

* $(-2)\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix} = 0$

A linear combination with nonzero weights that is equal to 0.

* We call $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$ linearly dependent.

DEFN (Linear Independence)

Let V be a vector space, and $v_1, \dots, v_n \in V$.

(1) $S = \{v_1, \dots, v_n\}$ is called linearly dependent if

$c_1 v_1 + \dots + c_n v_n = 0 \quad \exists c_1, \dots, c_n \in \mathbb{R}$ not all zero

(2) S is called linearly independent otherwise if

$c_1 v_1 + \dots + c_n v_n \neq 0 \quad \forall c_1, \dots, c_n \in \mathbb{R}$ not all zero

Suppose $v_1, \dots, v_n \in \mathbb{R}^m$.

$\{v_1, \dots, v_n\}$ is linearly dependent

$$\Longleftrightarrow$$

$c_1 v_1 + \dots + c_n v_n = 0 \quad \exists c_1, \dots, c_n$ not all zero

$$\Longleftrightarrow$$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

$\exists c_1, \dots, c_n$ not all zero

$$\Longleftrightarrow$$

$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} x = 0 \quad \exists x \neq 0 \in \mathbb{R}^n$

EXAMPLE

Verify whether the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

Solve $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} x = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[r_2 := r_2 - \cancel{2} r_1]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_3 := r_3 - r_2]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Consequently the only soln is

$$x_1 = x_2 = x_3 = 0.$$

The set is linearly independent. $\square$

REMARK

Suppose $\{v_1, \dots, v_n\}$ is linearly independent.

$$\text{span}\{v_1, \dots, v_j, v_{j+1}, \dots, v_n\} \neq \text{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$$

e.g.

$$\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\} = \mathbb{R}^2 \neq \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

Basis :

Basis for a vector space V is an optimal set B s.t. span $B = V$.

e.g.

NOT A BASIS FOR $\mathbb{R}^2$

$\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$ spans $\mathbb{R}^2$,

but it is not optimal in the sense that

$$\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

DEFN (Basis)

A subset B of a vector space V is called a basis if

(1) $\text{span } B = V$

(2) B is linearly independent.

EXAMPLES

① $\{1, x, x^2\}$ is a basis for

$$\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}.$$

(1) $\text{span}\{1, x, x^2\} = \mathbb{P}_2$

(2) $\{1, x, x^2\}$ is linearly independent

i.e.

$$c_1 + c_2 x + c_3 x^2 \equiv 0 \implies c_1 = c_2 = c_3 = 0$$

(consequently $\{1, x, x^2\}$ is minimal
e.g. $\text{span}\{1, x\} \neq \text{span}\{1, x, x^2\}$)

② 2×2 symmetric matrices

$$S^{2\times 2} = \{A \in \mathbb{R}^{2\times 2} : A^T = A\}$$

$$= \left\{ \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$$

$$= \left\{ a_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}$$

$$= _{(1)}\text{span} \underbrace{\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}}_{(2) \text{ linearly independent}}$$

$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{S}^{2\times 2}$.

LINEAR TRANSFORMATION

A transformation $T: V \to W$ is a function from a vector space V to another vector space W.

e.g.

$$T_1 : \mathbb{S}^{2\times 2} \longrightarrow \mathbb{R}^3$$

$$T_1\left(\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix} \right) = \begin{bmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{bmatrix}$$

$$T_2 : \mathbb{P}_2 \longrightarrow \mathbb{S}^{2\times 2}$$

$$T_2(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

<u>DEFN</u> (*Linear Transformation*)

A transformation $T: V \to W$ is called linear if

(1) $T(u) + T(v) = T(u+v) \quad \forall u, v \in V$

(2) $T(cu) = c\,T(u) \quad \forall u \in V, \forall c \in \mathbb{R}$

EXAMPLE

$$T_2(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

is linear.

(1) additivity

let $u = a_0 + a_1 x + a_2 x^2$, $v = b_0 + b_1 x + b_2 x^2$

$$T(u) + T(v) = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} + \begin{bmatrix} b_0 & b_1 \\ b_1 & b_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_0 + b_0 & a_1 + b_1 \\ a_1 + b_1 & a_2 + b_2 \end{bmatrix}$$

$$= T(u+v)$$

(2) scalar multiplicativity

let $u = a_0 + a_1 x + a_2 x^2$ and $c \in \mathbb{R}$

$$T(cu) = \begin{bmatrix} ca_0 & ca_1 \\ ca_1 & ca_2 \end{bmatrix}$$

$$= c \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix}$$

$$= c\,T(u)$$

But $T_1\left(\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1^2 \\ a_2^2 \\ a_3^2 \end{bmatrix}$ is not linear.

DEFN (Kernel & Range)

Let $T : V \to W$ be a transformation.

$$\text{Kernel}(T) := \{v \in V : T(v) = 0\}$$

$$\text{Range}(T) := \{T(v) \in W : v \in V\}$$

EXAMPLE

$$\text{Kernel}(T_2) = \{a_0 + a_1x + a_2x^2 : T_2(a_0 + a_1x + a_2x^2) = 0\}$$

$$= \left\{a_0 + a_1x + a_2x^2 : \begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} = 0\right\}$$

$$= \{0\}$$

$$\text{Range}(T_2) = \{T_2(a_0 + a_1x + a_2x^2) : a_0 + a_1x + a_2x^2 \in \mathbb{P}_2\}$$

$$= \left\{\begin{bmatrix} a_0 & a_1 \\ a_1 & a_2 \end{bmatrix} : a_0, a_1, a_2 \in \mathbb{R}\right\}$$

$$= \mathbb{S}^{2\times 2}$$

EXERCISE

Let $T : \mathbb{P}_3 \longrightarrow \mathbb{S}^{2\times 2}$ s.t.

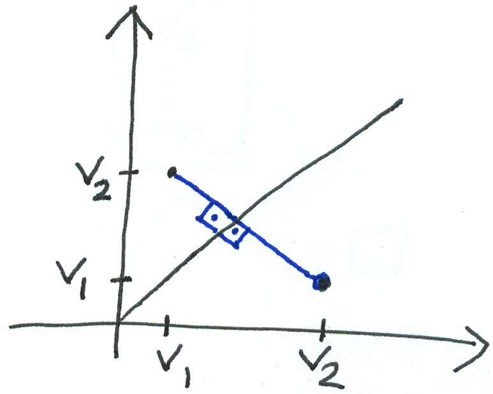
$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 & a_2 - a_1 \\ a_2 - a_1 & a_3 \end{bmatrix}$$

Find the kernel and range of T.

LINEAR TRANSFORMATIONS $\mathbb{R}^n \to \mathbb{R}^m$

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}$$



* Reflects about the line $v_2 = v_1$.
* linear

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = v_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow T(v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} v$$

THM

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a matrix $A \in \mathbb{R}^{m \times n}$ s.t.

$$T(v) = Av \qquad \forall v \in \mathbb{R}^n$$

PROOF

Let e_j denote the vector with all components zero except the jth one equal to one.

$$T(v) = T(v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$$

$$\text{ADDITIVITY} = T(v_1 e_1) + T(v_2 e_2) + \dots + T(v_n e_n)$$

$$\overset{\text{SCALAR MULTIPLICATIVITY}}{=} v_1 T(e_1) + v_2 T(e_2) + \dots + v_n T(e_n)$$

$$= \underbrace{\begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{bmatrix}}_{A \in \mathbb{R}^{m \times n}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{v}$$

$\square$

EXAMPLE

$$T : \mathbb{R}^3 \to \mathbb{R}^2$$

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 - 2v_2 + v_3 \\ -2v_1 + 3v_3 \end{bmatrix}$$

$$= \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} v$$

$$= \underbrace{\begin{bmatrix} 1 & -2 & 1 \\ -2 & 0 & 3 \end{bmatrix}}_{A \in \mathbb{R}^{2 \times 3}} v$$

Column and Null Space:

Consider $T : \mathbb{R}^n \to \mathbb{R}^m$

$$T(x) = Ax \quad \left(\text{any linear transformation}\right)$$

DEFN (Column & Null Spaces)

$$\text{Null}(A) := \text{Kernel}(T) = \{x \in \mathbb{R}^n : Ax = 0\}$$

$$\text{Col}(A) := \text{Range}(T) = \{Ax \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

EXAMPLE

Find the column and null spaces of

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\mathrm{Null}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \right\}$$

Need to solve $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned} \implies \begin{aligned} x_1 &= 2x_2 = 4x_3 \\ x_2 &= 2x_3 \end{aligned}$$

$\implies$ Any vector of form $\begin{bmatrix} 4x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$ is a soln.

$$\mathrm{Null}(A) = \left\{ \begin{bmatrix} 4x_3 \\ 2x_3 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_3 \in \mathbb{R} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\mathrm{Col}(A) = \left\{ \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \right\}$$

$$= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$= \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right\}$$

REMARKS

Let

$$A = [a_1 \ a_2 \ \cdots \ a_n] \in \mathbb{R}^{m\times n}$$

with $a_1, a_2, \dots, a_n \in \mathbb{R}^m$.

(1) The column space of A is also called the range of A and denoted by $Range(A)$.

(2)

$$Range(A) = Col(A) = span\{a_1, \dots, a_n\}$$