LECTURE 3

NORMS

A norm is used to measure the distances or lengths in a vector space.

DEFN (Norm)
Let V be a vector space. A norm on V is a function $\|\cdot\|: V \to \mathbb{R}$ satisfying the following properties.

(1) POSITIVITY:

$$\forall v \in V \quad \|v\| \geqslant 0, \text{ and}$$
$$\|v\| = 0 \iff v = 0$$

(2) HOMOGENEITY:

$$\forall v \in V, \ \forall \alpha \in \mathbb{R} \qquad \|\alpha v\| = |\alpha| \, \|v\|$$

(3) TRIANGLE INEQUALITY:

$$\forall v, w \in V \quad \|v+w\| \leqslant \|v\| + \|w\|$$

EXAMPLE
Let V be the space of continuous functions on $[a, b]$.

$$\underbrace{\|f\|_2}_{L_2\text{-norm}} = \int_a^b f^2(x)\,dx$$

e.g.

Let $f(x) = 1 + x$ on $[-1, 1]$

$$\|f\|_2 = \int_{-1}^{1} (1+x)^2 \, dx$$

$$= \int_{-1}^{1} 1 + 2x + x^2 \, dx$$

$$= x + x^2 + \frac{x^3}{3} \Big|_{-1}^{1} = \frac{8}{3}$$

COMMON VECTOR NORMS

Let V be $\mathbb{R}^n$ and $v \in \mathbb{R}^n$.

2-norm (Euclidean Norm)

$$\|v\|_2 = \sqrt{\sum_{j=1}^{n} |v_j|^2} = \sqrt{v^T v}$$

1-norm

$$\|v\|_1 = \sum_{j=1}^{n} |v_j|$$

∞-norm

$$\|v\|_\infty = \max_{j=1,\dots,n} |v_j|$$

p-norm

$$\|v\|_p = \sqrt[p]{\sum_{j=1}^{n} |v_j|^p}$$

(1-norm, 2-norm, ∞-norm are special cases)

e.g.

$$v = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \in \mathbb{R}^2$$

$$\|v\|_2 = \sqrt{(1)^2 + (-2)^2} = \sqrt{5}$$

$$\|v\|_1 = |1| + |-2| = 3$$

$$\|v\|_\infty = \max\{|1|, |-2|\} = 2$$

$$\|v\|_3 = \sqrt[3]{(1)^3 + |-2|^3} = \sqrt[3]{9}$$

THM

$\|\cdot\|_2$ is a vector norm.

LEMMA (Cauchy-Schwarz Inequality)

Let $w, v \in \mathbb{R}^n$. Then

$$|w^T v| \leq \|w\|_2 \, \|v\|_2$$

e.g.

$$\underbrace{\left| \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right|}_{3} \leq \underbrace{\left\| \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\|}_{\sqrt{5}} \cdot \underbrace{\left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\|}_{\sqrt{2}}$$

PROOF

(1) POSITIVITY

Obvious; squares are positive, and

$$v_j^2 = 0 \iff v_j = 0$$

(2) HOMOGENEITY

$$\|\alpha v\|_2 = \sqrt{\sum_{j=1}^{n} |\alpha v_j|^2}$$

$$= \sqrt{\alpha^2 \sum_{j=1}^{n} |v_j|^2}$$

$$= |\alpha| \sqrt{\sum_{j=1}^{n} |v_j|^2} = |\alpha| \, \|v\|_2$$

(3) TRIANGLE INEQUALITY

$$\|v+w\|_2 = \sqrt{(v+w)^T(v+w)}$$

$$= \sqrt{v^Tv + w^Tv + v^Tw + w^Tw}$$

$$= \sqrt{\|v\|_2^2 + 2v^Tw + \|w\|_2^2}$$

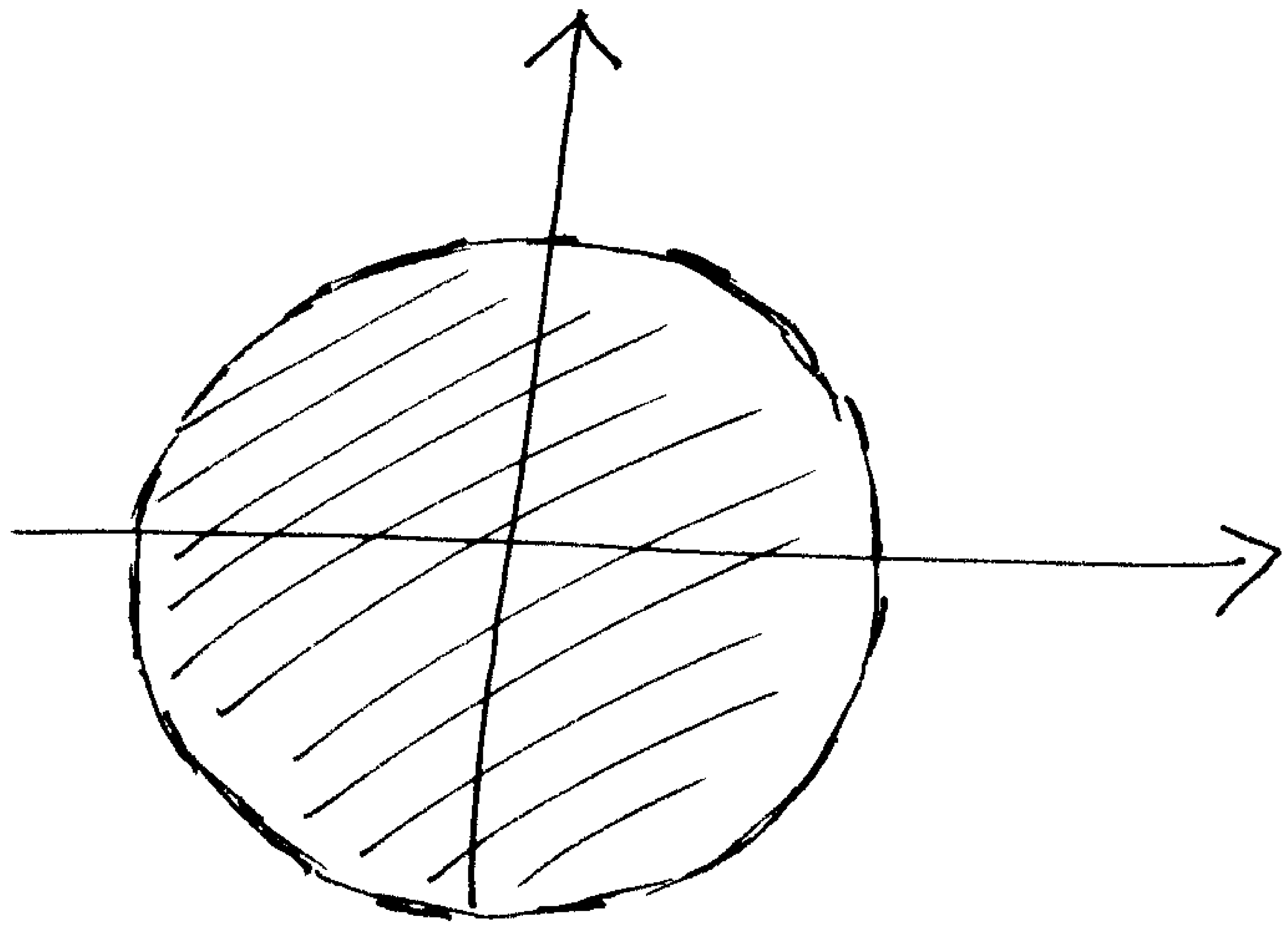
$$\left(\substack{\text{By} \\ \text{Cauchy-Schwarz}}\right) \leq \sqrt{\|v\|_2^2 + 2\|v\|_2\|w\|_2 + \|w\|_2^2}$$

$$= \sqrt{(\|v\|_2 + \|w\|_2)^2}$$

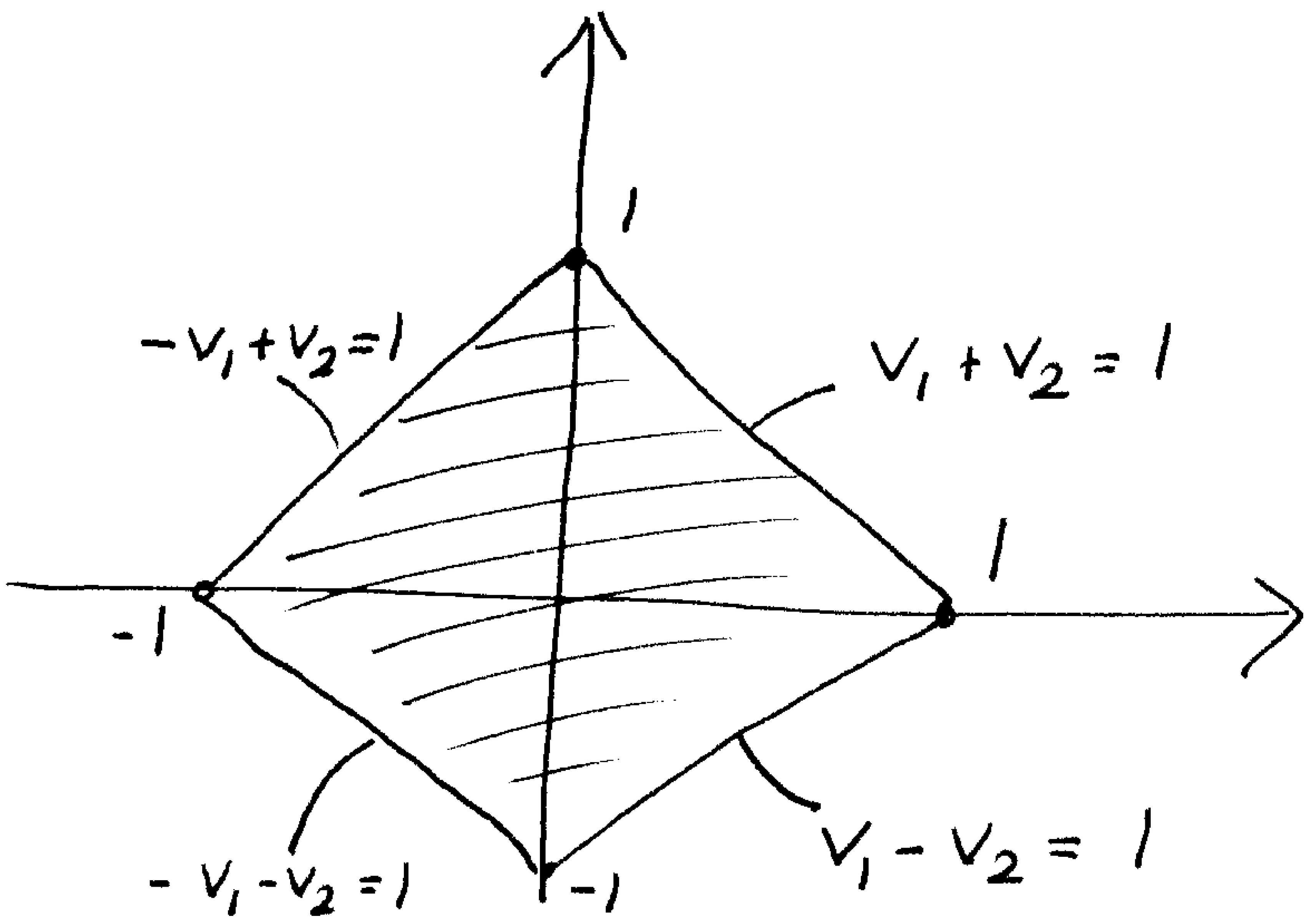
$$= \|v\|_2 + \|w\|_2$$

$\square$

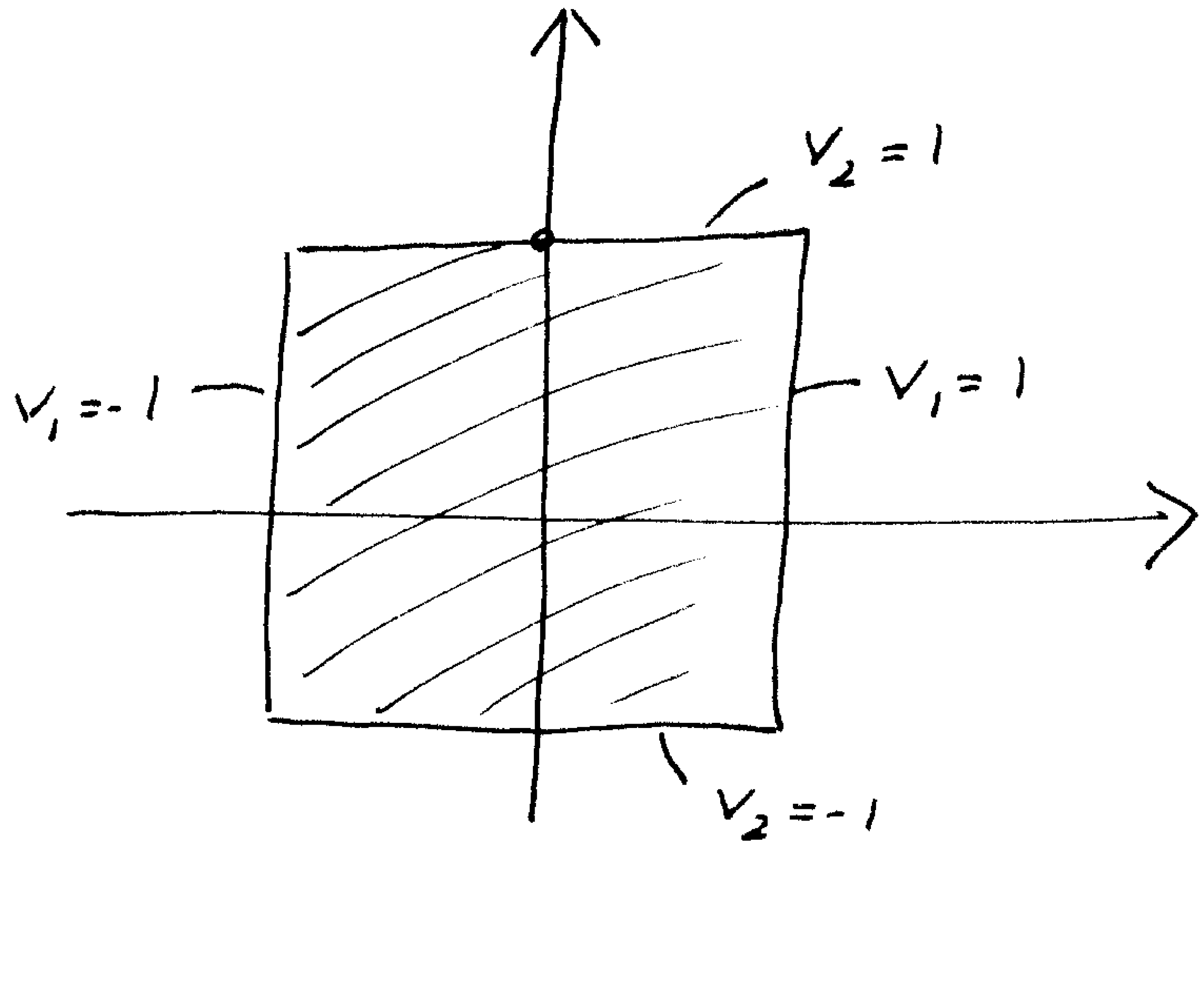
UNIT BALL IN $\mathbb{R}^2$ w.r.t. VARIOUS NORMS



$$B_2 = \{ v \mid \|v\|_2 \leqslant 1 \}$$
$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| \; v_1^2 + v_2^2 \leqslant 1 \right\}$$



$$B_1 = \{ v \mid \|v\|_1 \leqslant 1 \}$$
$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| \; |v_1| + |v_2| \leqslant 1 \right\}$$



$$B_\infty = \{ v \mid \|v\|_\infty \leqslant 1 \}$$
$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| \; \max\{|v_1|, |v_2|\} \leqslant 1 \right\}$$

COMMON MATRIX NORMS

Let V be $\mathbb{R}^{m \times n}$ and $A \in \mathbb{R}^{m \times n}$.

Frobenius Norm

$$\|A\|_F := \sqrt{\sum_{j=1}^{m} \sum_{k=1}^{n} a_{jk}^2}$$

e.g.

$$\left\| \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \right\|_F = \sqrt{1^2 + 2^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{18}$$

DEFN (Trace)

$$\text{Trace}(A) := \sum_{j=1}^{\min\{m,n\}} a_{jj}$$

THM

$$\|A\|_F = \sqrt{\text{Trace}(A^T A)}$$

PROOF

$$\|A\|_F = \sqrt{\sum_{k=1}^{n} \left(\sum_{j=1}^{m} a_{jk}^2 \right)}$$

$$= \sqrt{\sum_{k=1}^{n} \|a_k\|_2^2}$$

$$= \sqrt{\sum_{k=1}^{n} \underbrace{a_k^T a_k}_{\substack{(k,k)\text{th} \\ \text{entry of } A^T A}}} = \sqrt{\text{Trace}(A^T A)}$$

$\square$

2 – norm (Induced by vector 2-norm)

$$\|A\|_2 := \max_{\substack{x \in \mathbb{R}^n \\ \text{s.t.} \\ \|x\|_2 = 1}} \|Ax\|_2$$

e.g.

$$\left\| \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \right\|_2 = \max_{\substack{x \in \mathbb{R}^2 \\ \text{s.t.} \\ \|x\|_2 = 1}} \underbrace{\left\| \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2}_{\left\| \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 2x_2 \end{bmatrix} \right\|_2}$$

$$= \max_{\substack{x \in \mathbb{R}^2 \\ \text{s.t.} \\ \|x\|_2 = 1}} \sqrt{(2x_1 + x_2)^2 + (-x_1 + 2x_2)^2}$$

$$= \max_{\substack{x \in \mathbb{R}^2 \\ \text{s.t.} \\ \|x\|_2 = 1}} \sqrt{5x_1^2 + 5x_2^2}$$

$$= \sqrt{5}$$

1 – norm

$$\|A\|_1 := \max_{\substack{x \in \mathbb{R}^n \\ \text{s.t.} \\ \|x\|_1 = 2}} \|Ax\|_1$$

THM (Maximal Column Sum)

$$\|A\|_1 = \max_{j=1,\dots,n} \|a_j\|_1$$

PROOF

① Proof of $\|A\|_1 \leq \max\limits_{j=1,\dots,n} \|a_j\|_1$

Let $\|a_k\| := \max\limits_{j=1,\dots,n} \|a_j\|$.

For all $x \in \mathbb{R}^n$

$$\begin{aligned}
\|Ax\|_1 &= \Big\|\sum_{j=1}^{n} x_j a_j\Big\|_1 \\
\text{(By Triangle Inequality)} &\leq \sum_{j=1}^{n} \|x_j a_j\|_1 \\
\text{(By Homogeneity)} &= \sum_{j=1}^{n} |x_j| \, \|a_j\|_1 \\
&\leq \sum_{j=1}^{n} |x_j| \, \|a_k\|_1 \\
&= \|a_k\|_1
\end{aligned}$$

② Proof of $\|A\|_1 \geq \max\limits_{j=1,\dots,n} \|a_j\|_1$

Let $\|a_k\|_1 := \max\limits_{j=1,\dots,n} \|a_j\|_1$.

$$\|a_k\|_1 = \|Ae_k\|_1 \leq \max_{\substack{x \in \mathbb{R}^n \\ \text{s.t.} \\ \|x\|=1}} \|Ax\|_1 = \|A\|_1$$

$\square$

e.g.

$$\left\| \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \right\|_1 = \max \left\{ \left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|_1 , \left\| \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\|_1 \right\}$$

$$= 5$$

∞ - Norm

$$\|A\|_\infty := \max_{\substack{x \in \mathbb{R}^n \\ \text{s.t.} \\ \|x\|_\infty = 1}} \|Ax\|_\infty$$

THM (Maximal Row Sum)

$$\|A\|_\infty = \max_{j=1,\dots,m} \|\bar{a}_j^T\|_1$$

e.g.

$$\left\| \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \right\|_\infty = \max \left\{ \|[1 \ \ -2]^T\|_1 , \|[2 \ \ 3]^T\|_1 \right\}$$

$$= 5$$

Induced (p,q) - Norm

$$\|A\|_{p,q} := \max_{\substack{x \in \mathbb{R}^n \\ \text{s.t.} \\ \|x\|_q = 1}} \|Ax\|_p$$

$1, 2, \infty$ are special cases.

REMARKS

Norms on $\mathbb{C}^n$ and $\mathbb{C}^{m\times n}$ are defined similarly with the following exceptions.

* In all specific norms $|\cdot|$ denotes the modulus of a complex number.
* In all definitions v^T is replaced by v^*.

e.g.

Let $v \in \mathbb{R}^n$.

$$\|v\|_2 = \sqrt{v^* v}$$

$$= \sqrt{\sum_{j=1}^{n} |v_j|^2}$$

For instance

$$\left\| \begin{bmatrix} 1+2i \\ 1-3i \end{bmatrix} \right\|_2 = \sqrt{(\sqrt{5})^2 + (\sqrt{10})^2}$$

$$= \sqrt{15}$$