LECTURE 6

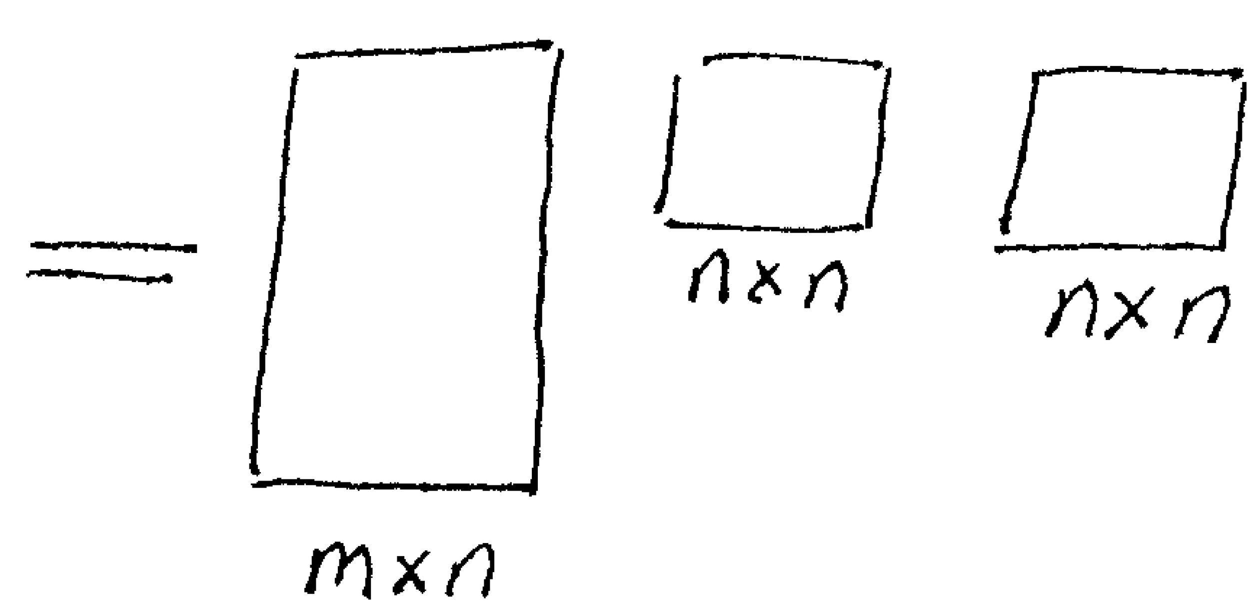
SINGULAR VALUE DECOMPOSITION (CONTINUES)

Rectangular Matrices

Let $A \in \mathbb{C}^{m\times n}$ with $m \geq n$. Then the (reduced) SVD is of the form (existence proof is similar to the square case)

$$A = \hat{U}\hat{\Sigma}\hat{V}^*$$

$\boxed{\text{REDUCED SVD}}$



$\hat{U} \in \mathbb{C}^{m\times n}$ with orthonormal columns (i.e. $\{u_1, \ldots, u_n\}$ is an orthonormal set)

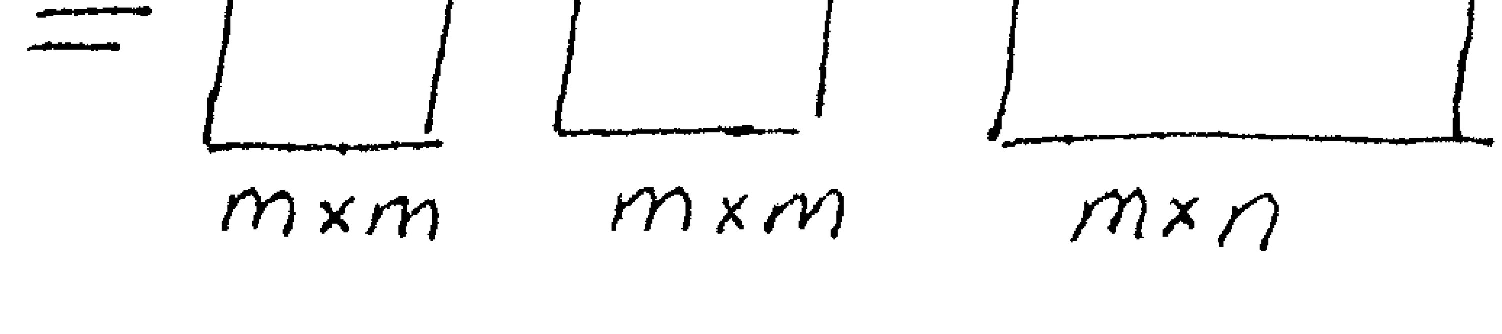
$\hat{\Sigma} \in \mathbb{R}^{n\times n}$ is diagonal with nonnegative entries

$\hat{V} \in \mathbb{C}^{n\times n}$ is unitary (i.e. $V^*V = I$)

REMARK

Reduced SVD for $A \in \mathbb{C}^{m\times n}$ with $m < n$.

$$A = \hat{U}\hat{\Sigma}\hat{V}^*$$



$\hat{U} \in \mathbb{C}^{m\times m}$ is unitary

$\hat{\Sigma} \in \mathbb{R}^{n\times n}$ is diagonal with nonnegative entries

$\hat{V} \in \mathbb{C}^{n\times m}$ with orthonormal columns
(i.e. $\{v_1, v_2, \dots, v_m\}$ is orthonormal)

EXAMPLE

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{6} & -3/\sqrt{11} \\ 1/\sqrt{6} & 1/\sqrt{11} \\ 2/\sqrt{6} & 1/\sqrt{11} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{12} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{V^T}$$

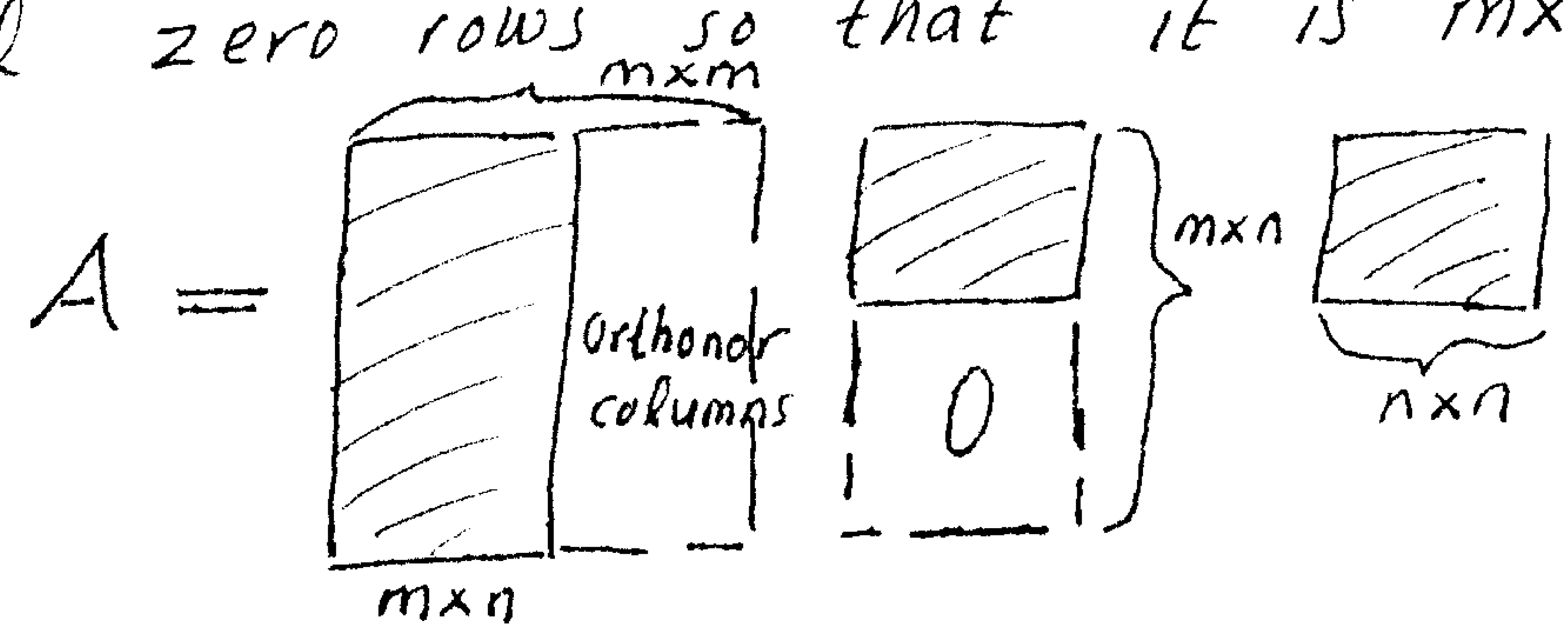
U (orthonormal columns) $U^T U = I$, but $UU^T \neq I$

V^T (orthogonal) $V^T V = V V^T = I$

Full SVD for Rectangular Matrices

Let $A \in \mathbb{C}^{m\times n}$ with $m \geqslant n$. Augment the factor $\hat{U}$ in the reduced SVD by additional columns so that it is square and ~~orthogonal~~ unitary. Augment the corresponding rows of $\hat{\Sigma}$ by additional zero rows so that it is $m\times n$.

$\boxed{\text{FULL SVD}}$



$$= U \Sigma V^*$$

$U \in \mathbb{C}^{m \times m}$ is unitary.

$\Sigma \in \mathbb{C}^{m \times n}$ is diagonal with nonnegative entries.

$V \in \mathbb{C}^{n \times n}$ is unitary.

EXAMPLE

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \left[\begin{array}{cc|c} 1/\sqrt{6} & -3/\sqrt{11} & 1/\sqrt{66} \\ 1/\sqrt{6} & 1/\sqrt{11} & 7/\sqrt{66} \\ 2/\sqrt{6} & 1/\sqrt{11} & -4/\sqrt{66} \end{array}\right] \begin{bmatrix} \sqrt{12} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Applications

① Column and Null Spaces

$\boxed{\text{Reduced SVD for } A \in \mathbb{C}^{m \times n}}$

$$A = [u_1 \; \cdots \; u_n] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^* \\ \vdots \\ v_n^* \end{bmatrix}$$

(*)

$$= [u_1 \; \cdots \; u_n] \begin{bmatrix} \sigma_1 v_1^* \\ \vdots \\ \sigma_n v_n^* \end{bmatrix} = \sum_{j=1}^{n} \sigma_j u_j v_j^*$$

EXAMPLE

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & -3/\sqrt{11} \\ 1/\sqrt{6} & 1/\sqrt{11} \\ 2/\sqrt{6} & 1/\sqrt{11} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= (\sqrt{12}) \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} + (0) \begin{bmatrix} -3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & +1/\sqrt{2} \end{bmatrix}$$

$$\text{Range}\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} \right) = \left\{ \sqrt{12} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} x}_{\text{scalar}} \;\middle|\; x \in \mathbb{R}^2 \right\}$$

$$= \left\{ \alpha \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

$$= \text{span} \left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for Range

(Let $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$)

$$\text{Null}\left(\begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} \right) = \left\{ \alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^2 \;\middle|\; \sqrt{12} \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \times (\alpha_1 v_1 + \alpha_2 v_2) = 0 \right\}$$

$$= \left\{ \alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^2 \;\middle|\; \sqrt{12}\, \alpha_1 \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix} \|v_1\|^2 = 0 \right\}$$

$$= \left\{ \alpha_2 v_2 \in \mathbb{R}^2 \;\middle|\; \alpha_2 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal basis for Null space.

THM (SVD for Null Space & Range)

Let $A \in \mathbb{C}^{m \times n}$ have an SVD as in $(*)$ with $\sigma_{j+1} = \sigma_{j+2} = \cdots = \sigma_n = 0$ and $\sigma_1 > \sigma_2 > \cdots > \sigma_j > 0$.

Then

(1) $\{u_1, \dots, u_j\}$ is an orthonormal basis in $\mathbb{C}^m$ for $\text{Range}(A)$,

(2) $\{v_{j+1}, \dots, v_n\}$ is an orthonormal basis in $\mathbb{C}^n$ for $\text{Null}(A)$.

PROOF

(1)

$$\begin{aligned}
\text{Range}(A) &= \{ Ax \mid x \in \mathbb{C}^n \} \\
&= \{ A(\alpha_1 v_1 + \cdots + \alpha_n v_n) \mid \alpha_1, \dots, \alpha_n \in \mathbb{C} \} \\
&= \{ \sigma_1 \alpha_1 u_1 + \cdots + \sigma_n \alpha_n u_n \mid \alpha_1, \dots, \alpha_n \in \mathbb{C} \} \\
&= \{ \sigma_1 \alpha_1 u_1 + \cdots + \sigma_j \alpha_j u_j \mid \alpha_1, \dots, \alpha_j \in \mathbb{C} \} \\
&= \text{span}\{u_1, \dots, u_j\}
\end{aligned}$$

Since the set $\{u_1, \dots, u_j\}$ is also orthonormal, it is an orthonormal basis for $\text{Range}(A)$.

(2)

$$\text{Null}(A) = \{x \in \mathbb{C}^n \mid Ax = 0\}$$

$$= \left\{x \in \mathbb{C}^n \,\middle|\, \sum_{k=1}^{n} \sigma_k u_k v_k^* x = 0\right\}$$

$$= \left\{\alpha_1 v_1 + \dots + \alpha_n v_n \in \mathbb{C}^n \,\middle|\, \sum_{k=1}^{n} \sigma_k u_k v_k^* (\alpha_1 v_1 + \dots + \alpha_n v_n) = 0\right\}$$

But

$$\sum_{k=1}^{n} \sigma_k u_k v_k^* (\alpha_1 v_1 + \dots + \alpha_n v_n) = \cancel{0}$$

$\left(\underline{\text{NOTE}}\quad v_k^* v_\ell = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}\right)$

$$\sum_{k=1}^{j} \sigma_k \alpha_k u_k = 0 \iff \alpha_1 = \dots = \alpha_j = 0$$

(SINCE $\sigma_k > 0$ $k = 1, \dots, j$ AND $\{u_1, \dots, u_j\}$ is linearly independent)

Consequently

$$\text{Null}(A) = \{\alpha_1 v_1 + \dots + \alpha_n v_n \mid \alpha_1 = \dots = \alpha_j = 0\}$$

$$= \text{span}\{v_{j+1}, \dots, v_n\}$$

Once again $\{v_{j+1}, \dots, v_n\}$ is orthonormal, meaning it is an orthonormal basis for Null(A).

$\square$

② 2-Norm and Frobenius Norm

Let $A \in \mathbb{C}^{m \times n}$ with (reduced) SVD

$$A = \hat{U} \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma} \hat{V}^*$$

$$\Longrightarrow \hat{U}^* A \hat{V} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

where $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n$.

By unitary invariance

$$\|A\|_2 = \|\hat{U}^* A \hat{V}\|_2 = \|\Sigma\|_2 = \sigma_1$$

$$\|A\|_F = \|\Sigma\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

<u>THM</u> (Norms and Singular Values)

$$\|A\|_2 = \sigma_1$$

$$\|A\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

<u>EXAMPLE</u>

$$\left\| \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 2 & -2 \end{bmatrix} \right\|_F = \sqrt{12}$$ (see example on page ②)