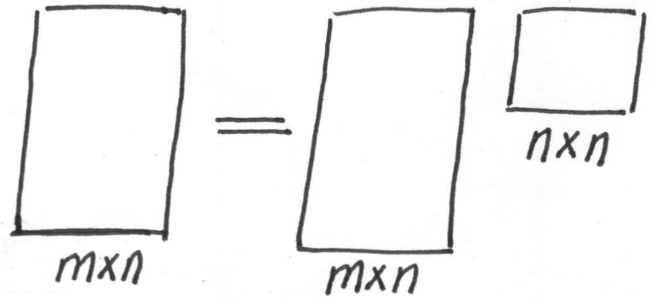
LECTURE 8

QR FACTORIZATION

Every matrix $A \in \mathbb{C}^{m \times n}$ (with $m \geq n$) has the reduced QR factorization of the form

$$A = \hat{Q}\hat{R}$$



$\hat{Q} \in \mathbb{C}^{m \times n}$ has orthonormal columns

$\hat{R} \in \mathbb{C}^{n \times n}$ is upper triangular

e.g.

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}}_{\hat{Q}} \underbrace{\begin{bmatrix} \sqrt{3} & 4\sqrt{3} \\ 0 & \sqrt{6} \end{bmatrix}}_{\hat{R}}$$

Given a reduced QR factorization

$$A = [q_1 \; q_2 \; \cdots \; q_n] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \ddots & r_{2n} \\ & & \ddots & \vdots \\ 0 & & & r_{nn} \end{bmatrix}$$

$\{q_1, q_2, \cdots, q_n\}$ is an orthonormal basis for Range(A) if $\hat{R}$ is invertible (that is $r_{jj} \neq 0$ for $j=1,\dots,n$)

since

$$
\begin{aligned}
y \in \text{Range}(A) &\Longrightarrow Ax = y \;\exists x \\
&\Longrightarrow \hat{Q}\hat{R}x = y \;\exists x \\
&\Longrightarrow \hat{Q}z = y \;\exists z \;(z = \hat{R}x) \\
&\Longrightarrow y \in \text{Range}(\hat{Q})
\end{aligned}
$$

and

$$
\begin{aligned}
y \in \text{Range}(\hat{Q}) &\Longrightarrow \hat{Q}x = y \;\exists x \\
&\Longrightarrow \hat{Q}\hat{R}\hat{R}^{-1}x = y \;\exists x \\
&\Longrightarrow \hat{Q}\hat{R}z = Az = y \;\exists z \;(z = \hat{R}^{-1}x) \\
&\Longrightarrow y \in \text{Range}(A)
\end{aligned}
$$

e.g. $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$ is an orthonormal basis for span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} \right\}$

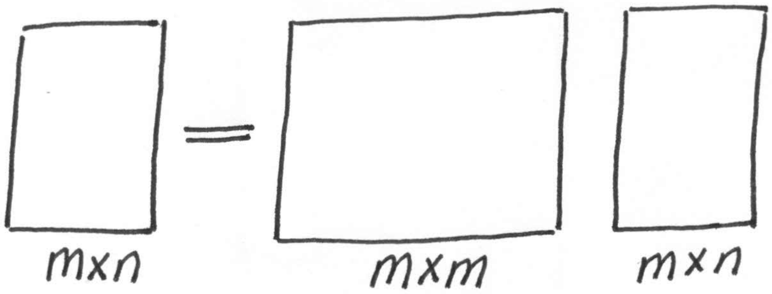
Similarly $\{q_1, q_2, \cdots, q_j\}$ is an orthonormal basis for span$\{a_1, a_2, \dots, a_j\}$ for $j = 1, \dots, n-1$ provided $\hat{R}$ is invertible.

e.g. $\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$ is an orthonormal basis for span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Full QR Factorization

Let $A \in \mathbb{C}^{m\times n}$ (with $m \geq n$). A factorization of the form

$$A = QR$$



$Q \in \mathbb{C}^{m\times m}$: unitary

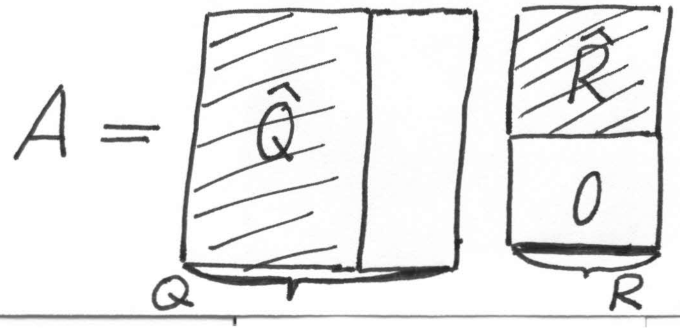
$R \in \mathbb{C}^{m\times n}$: upper triangular

e.g.

$$\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 4\sqrt{3} \\ 0 & \sqrt{6} \\ 0 & 0 \end{bmatrix}$$

Can be obtained from the reduced QR factorization by

* appending columns to $\hat{Q}$ to obtain a unitary matrix
* appending zero rows to $\hat{R}$



LECTURE 9

GRAM SCHMIDT ORTHOGONOLIZATION

Given

$$A = [a_1 \ a_2 \ \dots \ a_n] \in \mathbb{C}^{m\times n} \quad \begin{pmatrix}\text{with}\\ m \geq n\end{pmatrix}$$

This is a procedure to produce

(i) an orthonormal basis

$$\{q_1, q_2, \dots, q_n\}$$

for range $(A) = \text{span}\{a_1, \dots, a_n\}$

(ii) a reduced QR factorization

$$A = \hat{Q}\hat{R}$$

where

$$\hat{Q} = [q_1 \ q_2 \ \cdots \ q_n]$$

$\boxed{\text{Classical Gram-Schmidt}}$

$$\begin{pmatrix}\text{REDUCED}\\ \text{QR fact}\end{pmatrix} \quad A = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & r_{nn} \end{bmatrix}$$

$$\Longrightarrow$$

$$a_1 = r_{11} q_1$$
$$a_2 = r_{12} q_1 + r_{22} q_2$$
$$\vdots$$
$$a_n = r_{1n} q_1 + r_{2n} q_2 + \dots + r_{nn} q_n$$

In particular consider

$$(+)\ a_j = r_{1j} q_1 + r_{2j} q_2 + \dots + r_{jj} q_j .$$

Suppose

* $\{q_1, \dots, q_{j-1}\}$ is known,
* $r_{1j}, \dots, r_{jj}$ and q_j are sought.

Producing jth columns of $\hat{Q}$ and $\hat{R}$

(1) r_{ij} (for $i<j$) - multiply both sides of (+) by q_i^* from left

$$r_{ij} = q_i^* a_j , \quad i<j$$

(2) r_{jj} – scalar to normalize so that q_j is a unit vector

$$r_{jj} = \left\| a_j - \sum_{k=1}^{j-1} r_{kj} q_k \right\|$$

(3) q_j – component of a_j orthogonal to span$\{q_1, q_2, \dots, q_{j-1}\}$

$$q_j = \left(a_j - \sum_{k=1}^{j-1} r_{kj} q_k\right) \Big/ r_{jj}$$

EXAMPLE

Find a QR factorization for

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 5 \end{bmatrix}.$$

(1) First column of $\hat{Q}$ and $\hat{R}$.

$$r_{11} = \| a_1 \| = \sqrt{3}$$

$$q_1 = \frac{a_1}{r_{11}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(2) Second column of $\hat{Q}$ and $\hat{R}$.

$$r_{12} = a_2^T q_1 = [2 \ 5 \ 5] \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = 4\sqrt{3}$$

$$r_{22} = \| a_2 - r_{12} q_1 \| = \left\| \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} - 4\sqrt{3} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \right\|$$

$$= \left\| \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\|_2 = \sqrt{6}$$

$$q_2 = \frac{(a_2 - r_{12} q_1)}{r_{22}} = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Therefore

$$A = \begin{bmatrix} \underbrace{\begin{matrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{matrix}}_{q_1} & \underbrace{\begin{matrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{matrix}}_{q_2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 4\sqrt{3} \\ 0 & \sqrt{6} \end{bmatrix}$$

Order of computation

* At the jth iteration compute the jth cols of $\hat{R}$ and $\hat{Q}$.

  (1) First compute r_{ij} for $i<j$

  (2) Then compute r_{jj}

  (3) Finally compute q_j

<u>ALGORITHM</u> (Classical Gram-Schmidt)

* Given $A \in \mathbb{C}^{m\times n}$ (with $m \geqslant n$)
* Produce $\hat{Q} \in \mathbb{C}^{m\times n}$ (orthogonal) and $\hat{R} \in \mathbb{C}^{n\times n}$ (upper triangular) such that $A = \hat{Q}\hat{R}$

for $j=1,\dots,n$ (j : col #)

  for $i=1,\dots,j-1$

  $$r_{ij} = a_j^T q_i$$

  end

  $$r_{jj} = \left\| a_j - \sum_{k=1}^{j-1} r_{kj} q_k \right\|$$

  $$q_j = \left(a_j - \sum_{k=1}^{j-1} r_{kj} q_{kj}\right) / r_{jj}$$

end

Projector View

$$a_j = \underbrace{r_{1j} q_1 + r_{2j} q_2 + \dots + r_{(j-1)j} q_{j-1}}_{(a_j)_s} + \underbrace{r_{jj} q_j}_{w}$$

$(a_j)_s$: orthogonal projection of a_j onto $S = \operatorname{span}\{q_1, \dots, q_{j-1}\}$

That is

$$(a_j)_s = \underbrace{\begin{bmatrix} q_1 & \cdots & q_{j-1} \end{bmatrix}}_{\hat{Q}_j} \underbrace{\begin{bmatrix} q_1^* \\ \vdots \\ q_{j-1}^* \end{bmatrix}}_{\hat{Q}_j^*} a_j$$

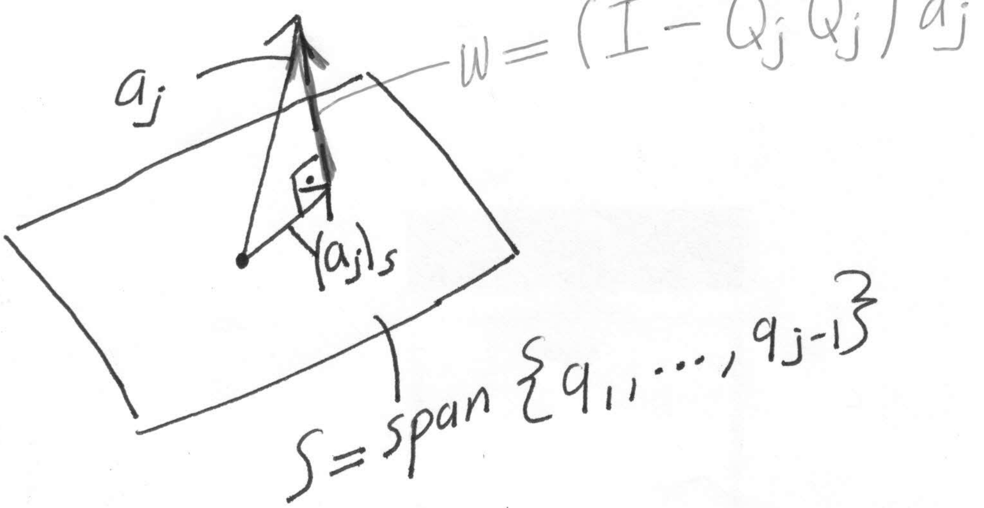
$$w = (I - \hat{Q}_j \hat{Q}_j^*) a_j$$

Summary

$$q_j = \frac{(I - \hat{Q}_j \hat{Q}_j^*) a_j}{\|(I - Q_j Q_j^*) a_j\|}$$

where

$I - \hat{Q}_j \hat{Q}_j^*$ is the orthogonal projector onto the subspace that is orthogonal to $\operatorname{span}\{q_1, \dots, q_{j-1}\}$



$$\boxed{q_j = \frac{w}{\|w\|}}$$

EXAMPLE

Consider $A = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 5 \end{bmatrix}$ with $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$w = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right) \right) \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}}_{(I - q_1 q_1^*)} \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

(orthogonal projector onto 2-dim subspace orthogonal to span$\{q_1\}$)

$$q_2 = w / \|w\| = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$