LECTURE 3

SINGULAR VALUE DECOMPOSITION (SVD)

Every matrix $A \in \mathbb{C}^{n \times n}$ has an SVD of the form

$$A = U \Sigma V^*$$

where

$U, V \in \mathbb{C}^{n \times n}$ are unitary matrices

Σ is a diagonal matrix with nonnegative real entries along the diagonal.

(If A is real, SVD is of the form
$$A = U \Sigma V^T$$
where $U, V \in \mathbb{R}^{n \times n}$ are orthogonal.)

EXAMPLE

$$\begin{bmatrix} 0.2 & 3.6 \\ 3.4 & 1.2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix}}_{V^T}$$

(Note that $U^T U = V^T V = I$)

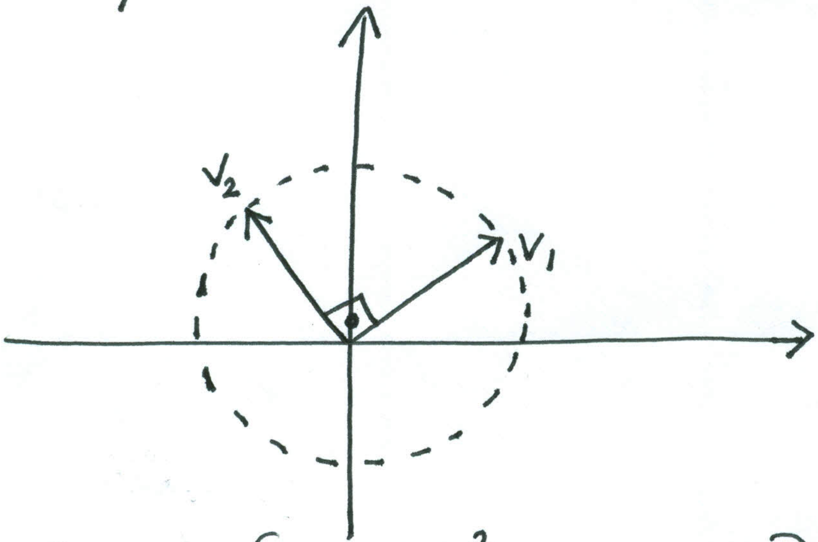
Consider the SVD for a 2x2 real matrix

$$A = \underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix}}_{V^T}$$

$u_1, u_2, v_1, v_2 \in \mathbb{R}^n$ s.t. $u_1 \perp u_2$ and $v_1 \perp v_2$

$\|v_1\|_2 = \|u_1\|_2 = \|u_2\|_2 = \|v_2\|_2 = 1$

Set of unit vectors in $\mathbb{R}^2$ (Unit Circle)



(Note
$\|\alpha_1 v_1 + \alpha_2 v_2\|_2 = 1$
$\Longleftrightarrow$
$\sqrt{\alpha_1^2 + \alpha_2^2} = 1$)

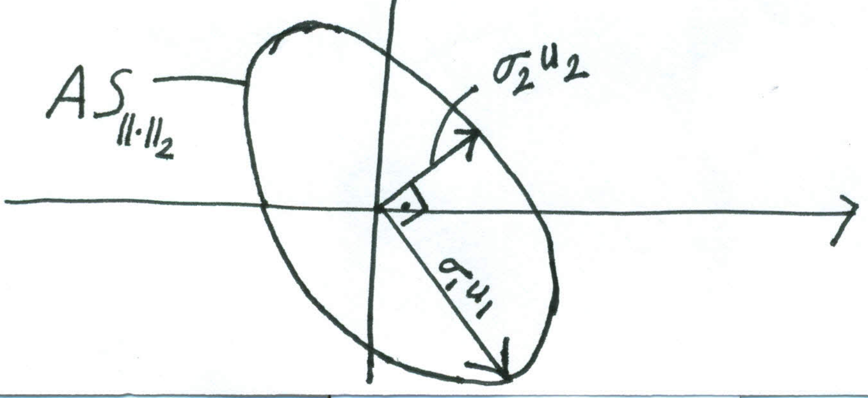
$$S_{\|\cdot\|_2} = \{v \in \mathbb{R}^2 : \|v\|_2 = 1\}$$

[redacted]

$$= \{\alpha_1 v_1 + \alpha_2 v_2 \in \mathbb{R}^2 : \alpha_1, \alpha_2 \in \mathbb{R} \text{ s.t. } \|\alpha_1 v_1 + \alpha_2 v_2\|_2 = 1\}$$

Image of $S_{\|\cdot\|_2}$ under the transformation

$x \longrightarrow Ax$



(Note that
$Av_1 = \sigma_1 u_1$
$Av_2 = \sigma_2 u_2$)

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2$$
$$= \alpha_1 \sigma_1 v_1 + \alpha_2 \sigma_2 v_2$$

i.e.

$$A S_{\|\cdot\|_2} = \{ Av \in \mathbb{R}^2 : \|v\|_2 = 1 \}$$
$$= \{ A(\alpha_1 v_1 + \alpha_2 v_2) \in \mathbb{R}^2 : \|\alpha_1 v_1 + \alpha_2 v_2\|_2 = 1 \}$$
$$= \{ \alpha_1 (\sigma_1 u_1) + \alpha_2 (\sigma_2 u_2) \in \mathbb{R}^2 : \|\alpha_1 v_1 + \alpha_2 v_2\| = 1 \}$$

In $\mathbb{R}^2$

$$(S_{\|\cdot\|_2})\ \text{Unit Circle with axes } v_1, v_2 \xrightarrow{A \in \mathbb{R}^{2\times 2}} \text{Ellipse with } (AS_{\|\cdot\|_2}) \text{ semi-axes } \sigma_1 u_1, \sigma_2 u_2$$

In $\mathbb{C}^n$

$$\text{Unit } n\text{-Sphere with axes } v_1, v_2, \dots, v_n \xrightarrow{A \in \mathbb{C}^{n\times n}} \text{Ellipsoid with semi-axes } \sigma_1 u_1, \dots, \sigma_n u_n$$

DEFINITION OF SINGULAR VALUES AND VECTORS

SVD for a 2x2 matrix

$$A = \underbrace{\begin{bmatrix} u_1 & u_2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix}}_{V^*}$$

$$\Longleftrightarrow$$

$$A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \text{ and } \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} A = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix}$$

$$\Longleftrightarrow$$

$$(1)\quad Av_1 = \sigma_1 u_1 \qquad u_1^* A = \sigma_1 v_1^*$$

and

$$(2)\quad Av_2 = \sigma_2 u_2 \qquad u_2^* A = \sigma_2 v_2^*$$

σ_1, σ_2 are called singular values

u_1, u_2 are called the associated left singular vectors

v_1, v_2 are called the associated right singular vectors

<u>DEFN</u> (Singular Values and Vectors)

Suppose that the equations

$$Av_i = \sigma_i u_i \quad \text{and} \quad u_i^* A = \sigma_i v_i^*$$

hold for some $u_i, v_i \in \mathbb{C}^n$ and $\sigma_i \geqslant 0$.

Then

(i) σ_i is called a singular value of A

(ii) u_i, v_i are called the left and right singular vectors associated with σ_i, respectively.

e.g.

$$\begin{bmatrix} 0.2 & 3.6 \\ 3.4 & 1.2 \end{bmatrix} \underbrace{\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}}_{v_1} = \underbrace{3\sqrt{2}}_{\sigma_1} \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}_{u_1}$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0.2 & 3.6 \\ 3.4 & 1.2 \end{bmatrix} = 3\sqrt{2} \begin{bmatrix} 0.6 & 0.8 \end{bmatrix}$$

σ_1 is called a singular value

$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $v_1 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$ are the associated left and right singular vectors

PROOF OF EXISTENCE OF SVD

Suppose that $\sigma_1 = \|A\|_2$ and $u_1, v_1 \in \mathbb{C}^n$ be such that

$$A v_1 = \sigma_1 u_1 .$$

(there exist such $u_1, v_1 \in \mathbb{C}^n$ since the maximization $\sigma_1 = \max\limits_{\|x\|=1} \|Ax\|$ is over a compact set and $\|Ax\|$ is continuous)

Now form the sets $\{u_1, \dots, u_n\}$ and $\{v_1, \dots, v_n\}$ that are orthonormal in $\mathbb{C}^n$.

Let $\tilde{U} = [u_1 \; u_2 \; \dots \; u_n]$ and $\tilde{V} = [v_1 \; v_2 \; \dots \; v_n]$. Then

$$\tilde{U}^* A \tilde{V} = \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix} [Av_1 \;\; Av_2 \; \dots \; Av_n]$$

$$= \begin{bmatrix} u_1^* \\ \vdots \\ u_n^* \end{bmatrix} [\sigma_1 u_1 \;\; Av_2 \; \dots \; Av_n]$$

$$= \begin{bmatrix} \sigma_1 & w^* \\ 0 & \tilde{A} \end{bmatrix}$$

where $w \in \mathbb{C}^n$ and $\tilde{A} \in \mathbb{C}^{(n-1)\times(n-1)}$.

Noting that $\|\tilde{U}^* A \tilde{V}\|_2 = \|A\|_2$ (By unitary invariance)

$$\sigma_1 = \|\tilde{U}^* A \tilde{V}\|_2 \geqslant \left\| \begin{bmatrix} \sigma_1 & w^* \\ 0 & \tilde{A} \end{bmatrix} \left(\begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \Big/ \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2 \right) \right\|_2$$

$$= \frac{\left\| \begin{bmatrix} \sigma_1 & w^* \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2}{\left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2}$$

$$\geqslant \frac{\sigma_1^2 + w^* w}{\left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2} = \left\| \begin{bmatrix} \sigma_1 \\ w \end{bmatrix} \right\|_2$$

Therefore $w = 0$ implying

$$\tilde{U}^* A \tilde{V} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \tilde{A} \end{bmatrix}.$$

Now the proof follows by induction on the size of A. Suppose for all $(n-1) \times (n-1)$ matrices there are singular value decompositions. In particular let

$$\tilde{A} = \hat{U} \hat{\Sigma} \hat{V}^*$$

be an SVD of $\tilde{A}$. Then

$$\tilde{U}^* A \tilde{V} = \begin{bmatrix} 1 & 0 \\ 0 & \hat{U} \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \hat{\Sigma} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \hat{V}^* \end{bmatrix}$$

$$\Longleftrightarrow$$

$$A = \underbrace{\tilde{U}\begin{bmatrix} 1 & 0 \\ 0 & \hat{U} \end{bmatrix}}_{(\text{Unitary})\, U} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \hat{\Sigma} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \hat{V}^* \end{bmatrix} \tilde{V}^*}_{V^*\, (\text{Unitary})}$$

Therefore A has an SVD.

$\square$

EXAMPLE

Consider

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

2-norm of A

$$\sigma_1 = \|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$= \max_{\|x\|_2=1} \left\| \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \right\|_2$$

$$= \max_{\|x\|_2=1} \sqrt{(x_1 - x_2)^2 + (x_1 + x_2)^2}$$

$$= \max_{\|x\|_2=1} \sqrt{2x_1^2 + 2x_2^2} = \sqrt{2}$$

Also

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}_{v_1} = \sqrt{2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{u_1}$$

Let $\tilde{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $\tilde{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\tilde{U}^T A \tilde{V} = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

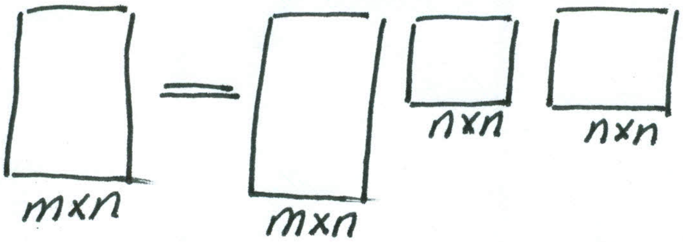
Therefore SVD of A

$$A = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\tilde{U}} \underbrace{\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{\tilde{V}^T}$$

SVD OF RECTANGULAR MATRICES

Every rectangular matrix $A \in \mathbb{C}^{m \times n}$ (with $m \geq n$) has a reduced SVD of the form

$$A = \hat{U} \hat{\Sigma} \hat{V}^*$$



where

$\hat{U} \in \mathbb{C}^{m \times n}$ has orthonormal columns

$\hat{V} \in \mathbb{C}^{n \times n}$ has orthonormal columns

$\hat{\Sigma} \in \mathbb{R}^{n \times n}$ is diagonal with nonnegative entries

EXAMPLE

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \\ 9 & 12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{14} & \frac{-5}{\sqrt{27}} \\ 2/\sqrt{14} & 1/\sqrt{27} \\ 3/\sqrt{14} & 1/\sqrt{27} \end{bmatrix}}_{\hat{U}} \underbrace{\begin{bmatrix} 5\sqrt{14} & 0 \\ 0 & 0 \end{bmatrix}}_{\hat{\Sigma}} \underbrace{\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}}_{\hat{V}}$$

Same definition for singular value - scalar σ satisfying

$$Av = \sigma u \quad \text{and} \quad u^* A = \sigma v^*$$

for some $u \in \mathbb{C}^m$, $v \in \mathbb{C}^n$ - applies.

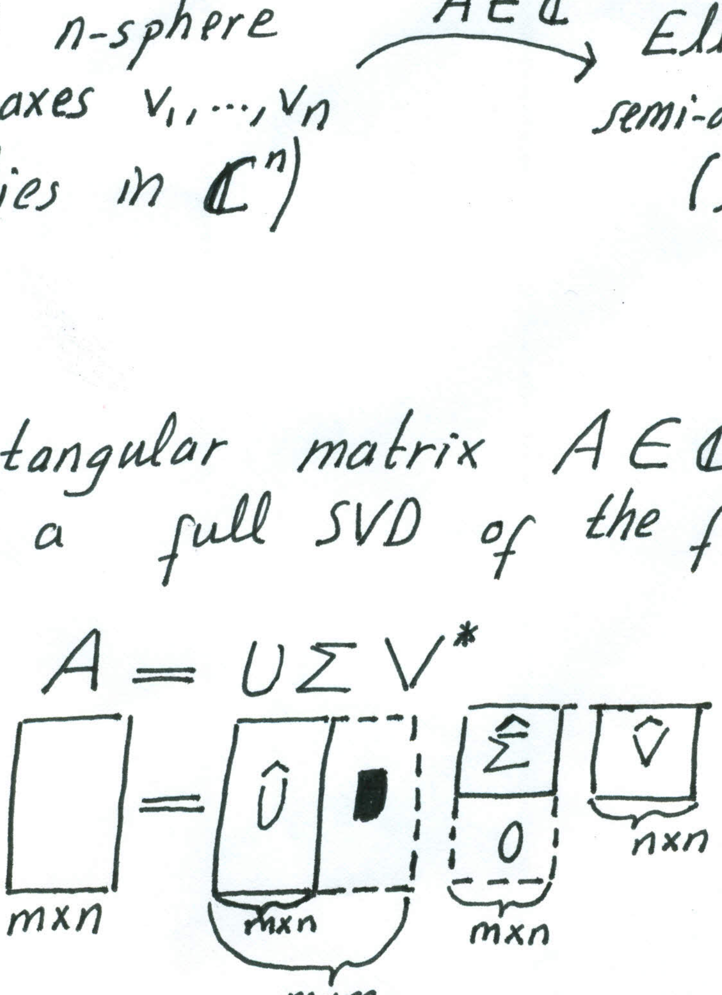
Same geometric interpretation

$$\begin{matrix} \text{Unit } n\text{-sphere} \\ \text{with axes } v_1, \dots, v_n \\ (\text{lies in } \mathbb{C}^n) \end{matrix} \xrightarrow{A \in \mathbb{C}^{m \times n}} \begin{matrix} \text{Ellipsoid with} \\ \text{semi-axes } \sigma_1 u_1, \dots, \sigma_n u_n \\ (\text{lies in } \mathbb{C}^m) \end{matrix}$$

applies.

Every rectangular matrix $A \in \mathbb{C}^{m \times n}$ (with $m \geq n$) also has a full SVD of the form

$$A = U \Sigma V^*$$



(Can be obtained from reduced SVD by appending cols to $\hat{U}$ so that U is unitary, appending $(m-n)$ 0 zero rows to $\hat{\Sigma}$)

where $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ are unitary
$\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with nonnegative entries