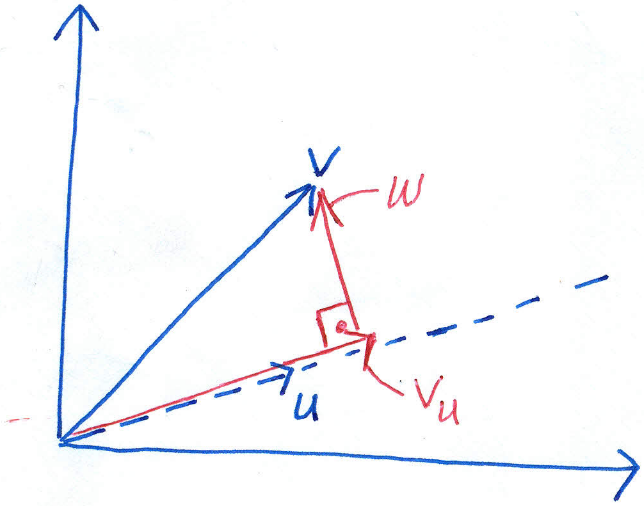
LECTURE 7

PROJECTORS

Projection onto a vector

Given $v \in \mathbb{C}^n$ and a unit vector $u \in \mathbb{C}^n$.



Find v_u such that

$v = v_u + w$ where

* $v_u = \overset{\text{scalar}}{\alpha} u$ is the orthogonal projection of v onto u
* $v_u \perp w$

Form an orthonormal basis for $\mathbb{C}^n$
$\{u, u_2, u_3, \ldots, u_n\}$.

Then

$$v = \underbrace{\alpha u}_{v_u} + \underbrace{\alpha_2 u_2 + \alpha_3 u_3 + \ldots + \alpha_n u_n}_{w}$$

Multiply v by u^* from left

$$u^* v = \alpha \underbrace{u^* u}_{1} + \alpha_2 \underbrace{u^* u_2}_{0} + \ldots + \alpha_n \underbrace{u^* u_n}_{0}$$

$$\Longrightarrow \alpha = u^* v \Longrightarrow \boxed{v_u = (u^* v)\, u}$$

EXAMPLE

Let $u = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$v_u = \left(\begin{bmatrix} 4/5 & -3/5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 4/25 \\ -3/25 \end{bmatrix}$$

$$w = v - v_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/25 \\ -3/25 \end{bmatrix}$$

$$= \begin{bmatrix} 21/25 \\ 28/25 \end{bmatrix}$$

Note that

$$v_u \perp w \quad \text{i.e.} \quad v_u^* w = \begin{bmatrix} 4/25 & -3/25 \end{bmatrix} \begin{bmatrix} 21/25 \\ 28/25 \end{bmatrix} = \underline{\underline{0}}$$

$\boxed{\text{Orthogonal Projectors}}$

Rearrange the expression for the orthogonal projection of v onto u.

$$v_u = (u^* v) u \;\cancel{= (v^* u) u}$$

$$= u (u^* v) = \underline{\underline{(u u^*) v}}$$

$P = u u^*$ is called the orthogonal projector onto span$\{u\}$.

e.g.

$$P = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \begin{bmatrix} 4/5 & -3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix}$$

is the orthogonal projector onto $\text{span}\left\{ \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix} \right\}$

For instance

$$v_u = Pv$$

$$\boxed{\text{orthogonal projection of } v \text{ onto } \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}} = \begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/25 \\ -3/25 \end{bmatrix}$$

P satisfies the following properties

(i) $P^2 = (uu^*)(uu^*) = uu^* = P$

(ii) $P^* = (uu^*)^* = uu^* = P$

<u>DEFN</u> (Orthogonal Projectors)

A matrix $P \in \mathbb{C}^{n \times n}$ is called an orthogonal projector if

(i) $P^2 = P$ and (ii) $P^* = P$.

More generally given a subspace S of $\mathbb{C}^n$.

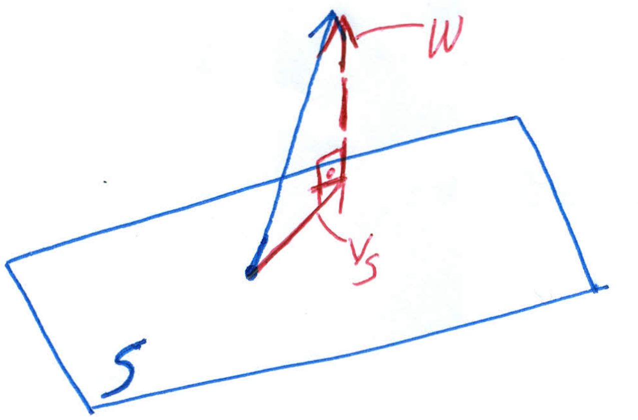
Let $\{u_1, u_2, \dots, u_m\}$ be an orthonormal basis for S (where $m \leqslant n$).

Orthogonal decomposition of $v \in \mathbb{C}^n$

$$v = v_S + w \qquad \text{where}$$

$v_S \in S$ (orthogonal projection of v onto S)

$w \perp S$ (that is $w^* u_j = 0$ for all $j \in [1, m]$)



<u>DEFN</u> (Orthogonal Projection)

Let $v \in \mathbb{C}^n$ and S be a subspace of $\mathbb{C}^n$. Then $v_S \in \mathbb{C}^n$ is called the orthogonal projection of v onto S if

(i) $v_S \in S$ and (ii) $(v - v_S) \perp v_S$.

REMARK

If $P \in \mathbb{C}^{n \times n}$ is an orthogonal projector and $v \in \mathbb{C}^n$, then Pv is the orthogonal projection of v onto range(P), i.e.

(i) $Pv \in \text{range}(P)$

(ii) $(v - Pv) \perp Pv$

since

$$
\begin{aligned}
(v-Pv)^* Pv &= v^*Pv - (Pv)^*Pv \\
&= v^*Pv - v^*P^*Pv \\
&= v^*Pv - v^*P^2v = 0
\end{aligned}
$$

EXAMPLE

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$ is an orthogonal projector.

* $$\underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}}_{P}$$

* $$\underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}^*}_{P^*} = \underbrace{\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}}_{P}$$

* Let $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Then $Pv = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (orthogonal projection of v onto range(P)) and $v - Pv = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Note $Pv \perp (v - Pv)$.

CONSTRUCTION OF AN ORTHOGONAL PROJECTOR

Let S be a subspace of $\mathbb{C}^n$
$\{u_1, u_2, \ldots, u_m\}$ be an orthonormal basis for S.

Form an orthonormal basis for $\mathbb{C}^n$

$$\{u_1, u_2, \ldots, u_m, u_{m+1}, \ldots, u_n\}$$

Any $v \in \mathbb{C}^n$ can be written of the form

$$v = \underbrace{\alpha_1 u_1 + \ldots + \alpha_m u_m}_{v_S} + \underbrace{\alpha_{m+1} u_{m+1} + \ldots + \alpha_n u_n}_{w}$$

Multiply with u_1^* from left to obtain

$$\alpha_1 = u_1^* v$$

Similarly

$$\alpha_j = u_j^* v \qquad \text{for } j = 2, \ldots, n.$$

Therefore

$$\begin{aligned} v_S &= \alpha_1 u_1 + \ldots\ldots + \alpha_m u_m \\ &= (u_1^* v) u_1 + \ldots\ldots + (u_m^* v) u_m \\ &= u_1 (u_1^* v) + \ldots + u_m (u_m^* v) \\ &= (u_1 u_1^* + \ldots + u_m u_m^*) v \end{aligned}$$

$P = u_1 u_1^* + \ldots + u_m u_m^*$ is the orthogonal projector onto S.

Orthogonal projector onto S

$$P = \sum_{j=1}^{m} u_j u_j^* = \underbrace{\begin{bmatrix} u_1 & u_2 & \dots & u_m \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_m^* \end{bmatrix}}_{U^*}$$

EXAMPLE

Find an orthogonal projector onto

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Note that

$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \right\}$$

is an orthonormal basis for S.

$$P = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

is an orthogonal projector onto S, i.e. for all $v \in \mathbb{C}^n$

(i) $Pv \in S$ and (ii) $(v - Pv) \perp Pv$

Oblique Projectors

Given two subspaces S_1, S_2 of $\mathbb{C}^n$ such that

$$(i)\ S_1 \cap S_2 = \{0\} \quad (ii)\ \dim(S_1) + \dim(S_2) = n$$

$$(\text{that is } S_1 \oplus S_2 = \mathbb{C}^n)$$

REMARK

$S_1 \oplus S_2$ stands for <u>direct-sum</u> of S_1 and S_2 defined as

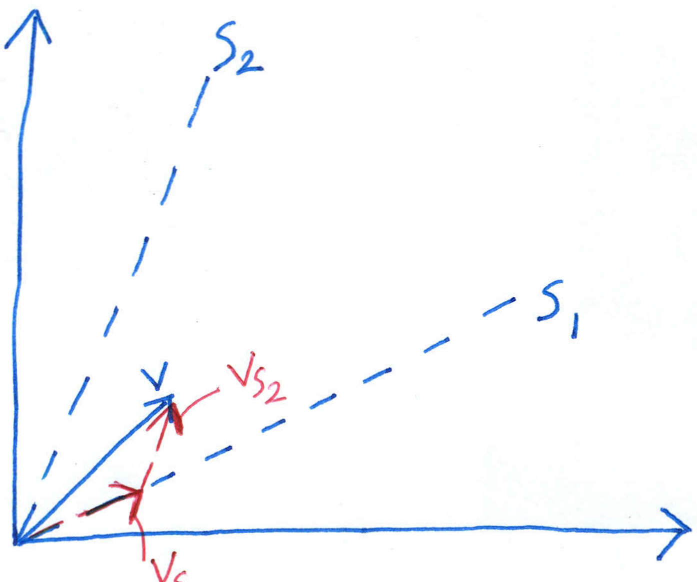
$$S_1 \oplus S_2 = \{v_1 + v_2 : v_1 \in S_1 \text{ and } v_2 \in S_2\}$$

e.g.

$$\text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\} \oplus \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\} = \mathbb{R}^2$$

since $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$ and any vector $v \in \mathbb{R}^2$ can be written of the form

$$v = \underbrace{\alpha_1 \begin{bmatrix}1\\2\end{bmatrix} + \alpha_2}_{\text{scalars}} \begin{bmatrix}2\\1\end{bmatrix}$$



Oblique decomposition of $v \in \mathbb{C}^n$

$$v = v_{S_1} + v_{S_2}$$

where

$$v_{S_1} \in S_1 \text{ and } v_{S_2} \in S_2$$

v_{S_1} : Oblique projection of v onto S_1 along S_2

EXAMPLE *

Let $S_1 = \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ and $S_2 = \text{span}\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$

Find the oblique projection of $v = \begin{bmatrix}1\\1\end{bmatrix}$ onto S_1 along S_2.

$$v = \alpha\begin{bmatrix}2\\1\end{bmatrix} + \beta\begin{bmatrix}1\\2\end{bmatrix} \iff v = \begin{bmatrix}2\alpha+\beta\\ \alpha+2\beta\end{bmatrix}$$

$$\iff \begin{aligned} 1 &= 2\alpha+\beta \\ 1 &= \alpha+2\beta \end{aligned}$$

$$\iff \alpha = \beta = 1/3$$

Then

$$v_{S_1} = \alpha\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2/3\\1/3\end{bmatrix}$$ is the projection onto S_1 along S_2.

DEFN (Oblique Projection)

Given two subspaces S_1, S_2 of $\mathbb{C}^n$ satisfying

(i) $S_1 \cap S_2 = \{0\}$ and (ii) $S_1 \oplus S_2 = \mathbb{C}^n$.

The (oblique) projection of $v \in \mathbb{C}^n$ onto S_1 along S_2 is the vector $v_{S_1} \in \mathbb{C}^n$ satisfying

(i) $v_{S_1} \in S_1$ and (ii) $(v - v_{S_1}) \in S_2$

DEFN (Projectors)

A matrix $P \in \mathbb{C}^{n\times n}$ is called a projector if $P^2 = P$.

REMARK

Suppose $P \in \mathbb{C}^{n \times n}$ is a projector. Then Pv is the projection of $v \in \mathbb{C}^{n \times n}$

* onto range(P)
* along range$(I-P)$

i.e.

First note that

$$\underbrace{\text{Range}(P)}_{S_1} \oplus \underbrace{\text{Range}(I-P)}_{S_2} = \mathbb{C}^n$$

(Since any $x \in \mathbb{C}^n$ can be written as $x = Px + (I-P)x$)

$$\text{Range}(P) \cap \text{Range}(I-P) = \{0\}$$

(since $z \in \text{Range}(P) \cap \text{Range}(I-P) \Longrightarrow$ for some $x, y \in \mathbb{C}^n$

$z = Px = (I-P)y$

$= P^2x = P(I-P)y$

$= (P-P^2)y$

$= 0$)

Additionally

(i) $Pv \in \text{Range}(P)$

(ii) $v - Pv \; (= (I-P)v) \in \text{Range}(I-P)$

EXAMPLE

$P = \begin{bmatrix} -1/3 & 2/3 \\ -2/3 & 4/3 \end{bmatrix}$ is a projector since

* $\begin{bmatrix} -1/3 & 2/3 \\ -2/3 & 4/3 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 \\ -2/3 & 4/3 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ -2/3 & 4/3 \end{bmatrix}$)

* onto range$(P) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

* along range$(I-P) = \text{range}\left(\begin{bmatrix} 4/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

REMARK

* If $P \in \mathbb{C}^{n \times n}$ is a projector, then so is $I-P$, i.e.,

$$(I-P)(I-P) = I - 2P + P^2$$
$$= I - 2P + P$$
$$= I - P$$

* $(I-P)v$ is the projection of v onto range$(I-P)$ along range(P).

EXAMPLE

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} - \underbrace{\begin{bmatrix} -1/3 & 2/3 \\ -2/3 & 4/3 \end{bmatrix}}_{P} = \begin{bmatrix} 4/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix}$$ is the projector

* onto range$(I-P) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} (= S_1)$
* along range$(P) = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

For instance

$$\begin{bmatrix} 4/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = v_{S_1}$$ (Compare this with example *)