SOLUTIONS TO MIDTERM 2

Q1.

(a) $XA = I \iff A^T X^T = I$

Find x_1 s.t.

$$A^T x_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For instance $x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Find x_2 s.t.

$$A^T x_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

For instance $x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Consequently

$$X^T = [x_1 \;\; x_2] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} \Longrightarrow A^L = X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

is a left-inverse.

(b) Projector onto Range(A) along Null(A^L) is given by

$$AA^L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Q2.

Recall that the computed $\tilde{y}$ in IEEE floating point arithmetic satisfies

$$\tilde{y} = (A + \delta A)\, x$$

for some δA s.t.

$$\frac{\|\delta A\|_1}{\|A\|_1} \leqslant n\, \epsilon_{mach}$$

Relative condition number

$$\kappa_\delta = \sup_{\|\delta A\|_1 \leqslant \delta} \frac{\|(A+\delta A)x - Ax\|_1 / \|Ax\|_1}{\|\delta A\|_1 / \|A\|_1}$$

$$= \frac{\|A\|_1 \|x\|_1}{\|Ax\|_1} \leqslant \|A\|_1 \|A^{-1}\|_1$$

$$\left(\text{SINCE } \|x\|_1 = \|A^{-1}Ax\|_1 \leqslant \|A^{-1}\|_1 \|Ax\|_1\right)$$

Recall

$$\frac{\|\delta y\|_1}{\|y\|_1} \leqslant \kappa_\delta \frac{\|\delta A\|_1}{\|A\|_1} \Longrightarrow \frac{\|\delta y\|_1}{\|y\|_1} \leqslant \cancel{\|A^{-1}\|_1 n \epsilon_{mach}} \; \|A\|_1 \|A^{-1}\|_1\, n\, \epsilon_{mach}$$

Q3.

Apply HH reflectors starting from right-most moving to left

(Assume $m \geq n$)

$$A \longrightarrow \underbrace{Q_1 A}_{A^{(1)}} = \begin{bmatrix} \times & \cdots & \times & 0 \\ \vdots & & \vdots & \vdots \\ \times & & \times & 0 \\ \times & & \times & \otimes \\ \times & & \vdots & \vdots \\ \times & \cdots & \times & \times \end{bmatrix} \; (n,n)$$

$$\longrightarrow \underbrace{Q_2 Q_1 A}_{A^{(2)}} = \begin{bmatrix} \times & \cdots & \times & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ \times & & \times & 0 & 0 \\ \times & & \times & \otimes & 0 \\ \times & & \times & \times & \times \\ \times & \cdots & \times & \times & \times \end{bmatrix} \; (n-1,n-1)$$

$$\vdots$$

$$\longrightarrow Q_n \cdots Q_2 Q_1 A = L$$

In particular Q_k introduces zeros at the kth stage on the $(n-k+1)$th column above the diagonal entry $(n-k+1,\ n-k+1)$.

$$Q_k = \begin{bmatrix} \hat{Q}_k & 0 \\ 0 & I_{k-1+m-n} \end{bmatrix}$$

$(k-1)+m-n \times (k-1)+m-n$ identity

③

where

$$\hat{Q}_k \in \mathbb{C}^{(n-k+1)\times(n-k+1)}$$

is HH reflector s.t

$$\hat{Q}_k \underbrace{a^{(k-1)}_{n-k+1}}_{\substack{(n-k+1)\text{th} \\ \text{column of } A^{(k-1)}}} = \underbrace{e_{n-k+1}}_{\substack{(n-k+1)\text{th} \\ \text{column of } I_{n-k+1}}} \| a^{(k-1)}_{n-k+1} \|$$

Specifically

$$\hat{Q}_k = I - 2uu^*$$

where

$$u = \frac{a^{(k-1)}_{n-k+1} - e_{n-k+1} \| a^{(k-1)}_{n-k+1} \|}{\left\| a^{(k-1)}_{n-k+1} - e_{n-k+1} \| a^{(k-1)}_{n-k+1} \| \right\|}$$

<u>Algorithm</u>

Input: $A \in \mathbb{C}^{m\times n}$ with $m \geq n$

Output: $Q \in \mathbb{C}^{m\times m}$ implicitly defined interms of HH vectors q_k) is unitary, $L \in \mathbb{C}^{m\times n}$ is lower triangular for $k = 1, \dots, n$

$$v = a_{1:n-k+1,\, n-k+1}$$

$$q_k = v - \|v\| e_{n-k+1} \quad \Big\} \; n-k+1 \text{ FLOPS}$$

$$q_k = q_k / \|q_k\| \quad \Big\} \; 1 \text{ FLOP}$$

$$\underbrace{A_{(1:n-k+1,\,1:n-k+1)} = A_{(1:n-k+1,\,1:n-k+1)} - 2q_k\left(q_k^* A_{1:n-k+1,\,1:n-k+1}\right)}_{4(n-k+1)^2 \text{ FLOPS}}$$

end

$L = A$

$$\text{Total \# Flops} = \sum_{k=1}^{n} 4(n-k+1)^2 + (n-k+2)$$

$$= \sum_{k=1}^{n} 4k^2 + (k+1)$$

$$= 4\,\frac{n(n+1)(2n+1)}{6} + \left(\frac{(n+2)(n+3)}{2} - 1\right)$$

$$= \frac{4n^3}{3} + O(n^2)$$

Q4.

(a)
$$\underset{x_1, x_0 \in \mathbb{R}}{\text{minimize}} \left\| \begin{bmatrix} l(s_1) - y_1 \\ l(s_2) - y_2 \\ l(s_3) - y_3 \end{bmatrix} \right\|_2 = \underset{x_1, x_0 \in \mathbb{R}}{\text{minimize}} \left\| \begin{bmatrix} 3x_1 + x_0 - 72 \\ 4x_1 + x_0 - 73 \\ 12x_1 + x_0 - 82 \end{bmatrix} \right\|_2$$

$$= \underset{\begin{bmatrix} x_1 \\ x_0 \end{bmatrix} \in \mathbb{R}^2}{\text{minimize}} \left\| \underbrace{\begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 12 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_0 \end{bmatrix} - \underbrace{\begin{bmatrix} 72 \\ 73 \\ 82 \end{bmatrix}}_{b} \right\|_2$$

(b) Normal Equation

$$A^* A x = A^* b$$

$$\Longrightarrow$$

$$\begin{bmatrix} 3 & 4 & 12 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 1 \\ 12 & 1 \end{bmatrix} x = \begin{bmatrix} 3 & 4 & 12 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 72 \\ 73 \\ 82 \end{bmatrix}$$

$$\Longrightarrow$$

$$\begin{bmatrix} 169 & 19 \\ 19 & 3 \end{bmatrix} x = \begin{bmatrix} 1492 \\ 227 \end{bmatrix}$$

Q5.

(a) The proof is by induction on the stage number k.

What we claim is at stage k (right before processing column k)

(*) $$L^{(k)} = \begin{bmatrix} \begin{bmatrix} & & 0 \\ \times & \ddots & \\ 0 & \times & 1 \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix}$$

$\hat{L}^{(k)} \in \mathbb{C}^{(k-1)\times(k-1)}$ lower triangular and tridiagonal

and

(**) $$U^{(k)} = \begin{bmatrix} \begin{bmatrix} \times & \times & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \times \\ 0 & & & \times \end{bmatrix} & \begin{bmatrix} 0 & & & \\ \vdots & & 0 & \\ 0 & & & \\ \times & 0 & \cdots & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} \times & \times & & 0 \\ \times & \times & \ddots & \\ & \ddots & \ddots & \times \\ 0 & & \times & \times \end{bmatrix} \end{bmatrix}$$

$u^{(k)}_{k,k}$

$u^{(k)}_{k+1,k}$

$\hat{U}^{(k)} \in \mathbb{C}^{(k-1)\times(k-1)}$ upper triangular and tridiagonal

$\tilde{U}^{(k)} \in \mathbb{C}^{(k-1)\times(n-k+1)}$ non-zero entry only at the lower right-most pos

$\tilde{\tilde{U}}^{(k)} \in \mathbb{C}^{(n-k+1)\times(n-k+1)}$ tridiagonal

Base Case

At stage $k=1$

$L^{(1)} = 0$ and $U^{(1)} = \tilde{\tilde{U}}^{(1)} = A$ is tridiagonal

so (*) and (**) hold

Inductive Case

Suppose (*) and (**) hold at stages $1, \dots, k-1$.

At stage k

* $l^{(k+1)}_{k+1,k} = \dfrac{u^{(k)}_{k+1,k}}{u^{(k)}_{k,k}}$ is introduced

$l^{(k+1)}_{k,k} = 1$ is set

No other entry of $L^{(k)}$ is modified

Consequently

$$L^{(k+1)} = \begin{bmatrix} \hat{L}^{(k+1)} & 0 \\ 0 & 0 \end{bmatrix}$$

$\hat{L}^{(k+1)} \in \mathbb{C}^{(k-1)\times(k+1)}$ lower triangular and tridiagonal

$*$

$$u^{(k+1)}_{k+1,k} = 0$$

$$u^{(k+1)}_{k+1,k+1} = u^{(k)}_{k+1,k+1} - \ell^{(k+1)}_{k+1,k} \cdot u^{(k)}_{k,k+1}$$

No other entry of $U^{(k)}$ is modified

Consequently

$$U^{(k+1)} = \begin{bmatrix} \hat{U}^{(k+1)} & \tilde{U}^{(k+1)} \\ 0 & \tilde{\tilde{U}}^{(k+1)} \end{bmatrix}$$

$\hat{U}^{(k+1)} \in \mathbb{C}^{k\times k}$ upper tri and tridiagonal

$\tilde{U}^{(k+1)} \in \mathbb{C}^{k\times(n-k)}$ only lower right-most entry nonzero

$\tilde{\tilde{U}}^{(k+1)} \in \mathbb{C}^{(n-k)\times(n-k)}$ tridiagonal

We deduce that (from $(*)$ and $(**)$)

$L = L^{(n+1)}$ is lower triangular and tridiagonal

$U = U^{(n+1)}$ is upper triangular and tridiagonal

(b) L and U are not tri-diagonal in general, if the rows are swapped.

EXAMPLE

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_2]{} \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[r_2 := -\frac{1}{2} r_1 + r_2]{} \begin{bmatrix} 2 & 1 & 3 \\ +1/2 & 5/2 & -3/2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[r_3 := -\frac{6}{5} r_2 + r_3]{} \begin{bmatrix} 2 & 1 & 3 \\ 1/2 & 5/2 & -3/2 \\ 0 & 6/5 & 14/5 \end{bmatrix}$$