Math-204 Differential Equations

Final Exam

Spring, 2019

NAME: Solutions

SECTION (check one box) : Section 1 (Inci 13:00) ☐
Section 2 (Kocyigit 13:00) ☐
Section 3 (Kocyigit 14:30) ☐

Instructions:

1. **Wait** for **signal** to begin.
2. **Write** your **name** in the space provided, and **check one box** to indicate which section of the course you belong to.
3. Please **turn off cell phones** or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must **justify your solutions** to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.
5. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
6. This exam is **closed book**. You may not use notes, or other external resource.
   It is a violation of the honor code to give or receive help on this exam.
7. In True/False questions circle true, false or if you think the given information is not sufficient for a conclusion then write N/A and explain. Unless the question states "No explanation needed" you should explain your answer in order to get credit.
8. If you use a theorem you should check (and write clearly) whether the conditions of the theorem hold.

Signature: ____________________

Problem	Score
1	12
2	11
3	10
4	10
5	10
6	9
7	9
8	9
Total	80

1. Given the system for unknown functions $u(t)$, $v(t)$

$$\begin{aligned} u' - 2u + 3v &= 0 \\ v' + 2v &= 0 \end{aligned}$$

(a) Write the above systems of equations in matrix form.

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Answer: $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{bmatrix} 2 & -3 \\ 0 & -2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

(b) Solve for $u(t)$ and $v(t)$

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Answer: $\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + 3c_2 e^{-2t} \\ 0 + 4c_2 e^{-2t} \end{pmatrix}$

Char eqn: $(\lambda-2)(-2-\lambda)=0 \Rightarrow \lambda_1 = 2 \quad \lambda_2 = -2$

$\lambda_1 = 2 \quad \begin{bmatrix} 0 & -3 \\ 0 & -4 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -2 \quad \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow 4a - 3b = 0 \Rightarrow v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

(c) Write the general solution of

$$\boldsymbol{x}' = A\boldsymbol{x},$$

in terms of real-valued functions where A is 3×3 matrix with the following eigenvalue - eigenvector pairs:

$$\lambda_1 = 1 - i, \quad \boldsymbol{v}_1 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_2 = 1 + i, \quad \boldsymbol{v}_2 = \begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_3 = 2, \quad \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

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Answer: $x(t) = c_1 e^{t} \begin{pmatrix} \sin t \\ 0 \\ \cos t \end{pmatrix} + c_2 e^{t} \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

From $\lambda_3 = 2$ soln is $e^{2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

From $\lambda_1 = 1 - i$ $\quad z(t) = e^{(1-i)t} \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} = e^{t} \begin{pmatrix} i(\cos t - i\sin t) \\ 0 \\ \cos t - i \sin t \end{pmatrix} = e^{t} \begin{pmatrix} \sin t + i\cos t \\ 0 \\ \cos t - i\sin t \end{pmatrix}$

$= e^{t} \begin{pmatrix} \sin t \\ 0 \\ \cos t \end{pmatrix} + i e^{t} \begin{pmatrix} \cos t \\ 0 \\ -\sin t \end{pmatrix}$

two more soln.

2. Suppose $P(t)$ is a $2x2$ matrix valued continuous function on $t \in \mathbb{R}$ and $\boldsymbol{x}_0$ is a non-zero constant vector. Given the equations

$$\boldsymbol{x}' = P(t)\boldsymbol{x} \qquad (1)$$

$$\boldsymbol{x}' = P(t)\boldsymbol{x} + \boldsymbol{x}_0 \qquad (2)$$

the following are known

- $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ are solutions to (1) with $\boldsymbol{u}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{u}(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{v}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}(2) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$
- $\boldsymbol{w}(t)$ and $\boldsymbol{z}(t)$ are solutions to (2) with $\boldsymbol{w}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\boldsymbol{w}(2) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

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(a) [Yes/No] Is there a constant c such that $c\boldsymbol{w} - \boldsymbol{z}$ is a solution to (1). If so then find c otherwise explain.

Substitute: $(cw-z)' = P(cw-z) \Leftrightarrow c(w'-Pw) = (z'-Pz)$

$\Leftrightarrow cx_0 = x_0 \Leftrightarrow \boxed{c=1}$ since $x_0 \neq 0$

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(b) [Yes/No] Is there a constant d such that $d\boldsymbol{w} - \boldsymbol{z}$ is a solution to (2). If so then find d otherwise explain.

Substitute: $(dw-z)' = P(dw-z) + x_0 \Leftrightarrow d(w'-Pw) = (z'-Pz) + x_0$

$\Leftrightarrow dx_0 = x_0 + x_0 \Leftrightarrow \boxed{d=2}$

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(c) [Yes/No] Let $\boldsymbol{x}(t)$ be a solution to (1) with $\boldsymbol{x}(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Is it possible to calculate $\boldsymbol{x}(2)$ from the given information? If so find $\boldsymbol{x}(2)$, otherwise explain.

u and v are linearly independent. Therefore gen. soln. is:

$x(t) = c_1 u(t) + c_2 v(t)$

evaluate at $t=0$: $x(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow$ $c_1 + c_2 = 5$, $c_1 = 4$, $c_2 = 1$

$\Rightarrow x(t) = 4u(t) + v(t)$

$$\boxed{x(2) = 4u(2) + v(2) = \begin{pmatrix} 4 \\ 6 \end{pmatrix}}$$

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3. Write the general solution of

$$x' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} x,$$

in terms of real-valued functions.

Answer: $x(t) = c_1 e^{2t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{2t}\begin{pmatrix}t\\1\end{pmatrix}$

Char eqn: $(\lambda-2)^2 = 0$ $\lambda = 2$ repeated root.

eigenvectors: $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = 0 \Rightarrow b = 0 \Rightarrow v_1 = \begin{pmatrix}1\\0\end{pmatrix}$

only one linearly indep. eigenvector

$\boxed{e^{2t}\begin{pmatrix}1\\0\end{pmatrix} \text{ is a soln}}$

gen. eigenvector $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow b = 1 \Rightarrow w = \begin{pmatrix}0\\1\end{pmatrix}$

$\boxed{te^{2t}\begin{pmatrix}1\\0\end{pmatrix} + e^{2t}\begin{pmatrix}0\\1\end{pmatrix} \text{ is another soln.}}$

$= e^{2t}\begin{pmatrix}t\\1\end{pmatrix}$

Alternatively you can compute $e^{\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}t}$.

4. Find the eigenvalues and eigenfunctions of the given BVP

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(\pi) = 0$$

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Answer: $\lambda_n = \left(\frac{n}{2}\right)^2 \quad y_n(x) = \sin\left(\frac{nx}{2}\right), \quad n = 1, 3, 5, \ldots$

$\boxed{\text{Case } \lambda > 0}$ $\quad \lambda = \mu^2 \quad y'' + \mu^2 y = 0$

gen soln. $\quad y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

B.C. $\begin{cases} 0 = c_1 \cdot 1 + 0 & \Rightarrow c_1 = 0 \\ 0 = c_1 \mu(-\sin(\mu x)) + c_2 \mu \cos(\mu x) & \Rightarrow c_2 \mu \cos(\mu \pi) = 0 \end{cases}$

$\Rightarrow \mu = \frac{n}{2}$ where $n = 1, 3, 5, \ldots$

$\lambda_n = \left(\frac{n}{2}\right)^2 \quad y_n(x) = \sin\left(\frac{nx}{2}\right) \quad n = 1, 3, 5, \ldots$

$\boxed{\text{Case } \lambda = 0}$ $\quad y'' = 0 \Rightarrow y(x) = ax + b$

BC: $\begin{cases} 0 = b \\ 0 = a \end{cases}$ $\quad a = b = 0 \quad$ no eigenvalue

$\boxed{\text{Case } \lambda < 0}$ $\quad \lambda = -\mu^2 \quad y'' - \mu^2 y = 0$

gen soln $\quad y(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$ $\quad$ [alternatively $c_1 e^{\mu x} + c_2 e^{-\mu x}$]

BC: $\begin{cases} 0 = c_1 + 0 & \Rightarrow c_1 = 0 \\ 0 = c_1 \mu \cancel{\sinh(\mu\pi)} + c_2 \mu \cosh(\mu \pi) & \Rightarrow c_2 = 0 \end{cases}$ $\quad$ no eigenvalues

5. Given

$$f(x) = 2x, \qquad -1 \le x < 1; \qquad f(x+2) = f(x).$$

$\boxed{10}$

Calculate the Fourier Series of $f(x)$.

Answer:	$\displaystyle\sum_{n=1}^{\infty} \frac{-4\cos(n\pi)}{n\pi}\sin(n\pi x)$ or $\displaystyle\sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin(n\pi x)$

$L=1$ f is odd $\Rightarrow$ $a_0 = 0$ $a_n = 0$ $\forall n$

$$b_n = \frac{2}{1}\int_0^1 2x\sin(n\pi x)\,dx = \frac{-4x\cos(n\pi x)}{n\pi}\Big|_0^1 + \int_0^1 \frac{4\cos(n\pi x)}{n\pi}\,dx$$

$$= \frac{-4\cos(n\pi)}{n\pi} + \cancel{\frac{4\sin(n\pi x)}{(n\pi)^2}\Big|_0^1}$$

6. Solve the following heat conduction problem.

$$\begin{aligned} u_{xx} &= u_t, & 0<x<1,\ t>0 \\ u(0,t) &= 0, & t\geq 0 \\ u(1,t) &= 0, & t\geq 0 \\ u(x,0) &= 2x & 0<x<1 \end{aligned}$$

Answer: $u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} e^{-n^2\pi^2 t} \sin(n\pi x)$

$\boxed{9}$

Separable: $u(x,t) = X(x)\cdot T(t) \Rightarrow X''T = XT'$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \Rightarrow \begin{array}{l} X''+\lambda X = 0 \\ T'+\lambda T = 0 \end{array}$$

BC: $\begin{cases} 0 = X(0)\,T(t) \\ 0 = X(1)\,T(t) \end{cases} \Rightarrow \begin{array}{l} X(0)=0 \\ X(1)=0 \end{array}$

ODE-x

$$\boxed{\begin{array}{l} X''+\lambda X = 0 \\ X(0)=0 \\ X(1)=0 \end{array}}$$

ODE-t

$$\boxed{T' = -\lambda T}$$

$T(t) = e^{-\lambda t}$

$T_n(t) = e^{-n^2\pi^2 t}$

$L=1$ soln one:

$n=1,2,3,\ldots$ $\lambda_n = \frac{n^2\pi^2}{1}$ $X_n(x) = \sin(n\pi x)$

Set: $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$

IC: $\sum_{n=1}^{\infty} c_n \sin(n\pi x) = u(x,0) = 2x = \sum_{n=1}^{\infty} \frac{+4(-1)^{n+1}}{n\pi} \sin(n\pi x)$ by previous problem

$\Rightarrow c_n = \frac{4(-1)^{n+1}}{n\pi}$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi x)\, e^{-n^2\pi^2 t}.$$

9

7. Solve for $y(t)$ using Laplace transform

$$y'' - 10y' + 9y = 5t, \qquad y(0) = -1,\ y'(0) = 2$$

Answer: $y(t) = \dfrac{31e^{9t}}{81} - 2e^{t} + \dfrac{5t}{9} + \dfrac{50}{81}$

$$\mathcal{L} \Rightarrow \quad s^2 Y + s - 2 - 10sY - 10 + 9Y = \frac{5}{s^2}$$

$$(s^2 - 10s + 9)Y = \frac{5}{s^2} + 12 - s$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{5}{s^2} \cdot \frac{1}{(s-1)(s-9)}\right\}}_{P(t)} + \underbrace{\mathcal{L}^{-1}\left\{\frac{12-s}{(s-1)(s-9)}\right\}}_{Q(t)}$$

$$Q(t) = \mathcal{L}^{-1}\left\{\frac{12-s}{(s-1)(s-9)}\right\} \longrightarrow \mathcal{L}^{-1}\left\{\frac{3/8}{s-9} + \frac{-11/8}{s-1}\right\} = \frac{3e^{9t}}{8} - \frac{11e^{t}}{8}$$

$$\left.\begin{aligned} A + B &= -1 \\ -A - 9B &= 12 \end{aligned}\right\} \quad A = 3/8 \quad B = -11/8$$

$$P(t) = \mathcal{L}^{-1}\left\{\frac{5}{s^2}\left[\frac{1/8}{s-9} - \frac{1/8}{s-1}\right]\right\} = 5t * \left[\frac{e^{9t}}{8} - \frac{e^{t}}{8}\right] = \frac{5}{8}\left(t * e^{9t} - t * e^{t}\right)$$

$$t * e^{\alpha t} = \int_0^t \tau e^{\alpha(t-\tau)}\,d\tau = e^{\alpha t}\left[\left.\frac{\tau e^{-\alpha\tau}}{-\alpha}\right|_0^t + \int_0^t \frac{e^{-\alpha\tau}}{\alpha}\,d\tau\right]$$

$$= e^{\alpha t}\left[\frac{-te^{-\alpha t}}{\alpha} - \frac{e^{-\alpha t}}{\alpha^2} + \frac{1}{\alpha^2}\right] = -\frac{t}{\alpha} - \frac{1}{\alpha^2} + \frac{e^{\alpha t}}{\alpha^2}$$

thus

$$P(t) = \frac{5}{8}\left[\frac{-t}{9} - \frac{1}{81} + \frac{e^{9t}}{81} + t + 1 - e^{t}\right] = \frac{5e^{9t}}{8 \cdot 81} - \frac{5e^{t}}{8} + \frac{5t}{9} + \frac{50}{81}$$

$$y(t) = Q(t) + P(t) = \frac{31e^{9t}}{81} - 2e^{t} + \frac{5t}{9} + \frac{50}{81}$$

8. Let $p(t), q(t), f(t)$ be all continuous functions on $t \in \mathbb{R}$

$$L[y] = y'' + py' + qy.$$

Consider the differential equations for $y(t)$

$$L[y] = 0 \qquad (1)$$

$$L[y] = f(t) \qquad (2)$$

[T / F] If u_1, u_2, u_3 are solutions to (2) on $\mathbb{R}$ then Wronskian $W(u_1 - u_3, u_2 - u_3)(t)$ is either 0 or never achieves 0 for all $t \in \mathbb{R}$.

3

explain or give counter example:

$u_1 - u_3$ and $u_2 - u_3$ are both solutions of (1)
Thus W is either 0 or never 0 on $t \in \mathbb{R}$
by Abel's theorem

[T / F] If v_1, v_2 are solutions to (1) such that $v_1(0) = v_2(0)$ then Wronskian $W(v_1, v_2)(t)$ is 0 for all $t \in \mathbb{R}$.

3

explain or give counter example: It is sufficient to check W at one point, say 0:

$$w(v_1 v_2)(0) = \begin{vmatrix} v_1(0) & v_2(0) \\ v_1'(0) & v_2'(0) \end{vmatrix} = v_1(0)\left[v_2'(0) - v_1'(0)\right]$$

might or might not be 0.

[T / F] If u_1, u_2 are solutions to (2) on $\mathbb{R}$ then Wronskian $W(u_1, u_2)(t)$ can achieve both 0 and non-zero values.

3

explain or give counter example: Note: [Abel's theorem does not apply to non-homogeneous (Thm 32.7)]

Counter example

$y'' = 2$

$u_1 = t^2$

$u_2 = t^2 + 1$

$$W = \begin{vmatrix} t^2 & t^2+1 \\ 2t & 2t \end{vmatrix} = -2t$$

Table 1: **Laplace Transform Table.**

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s>0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s>0$
$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$u_c(t)$	$\frac{e^{-cs}}{s}$, $s>0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f'(t)$	$sF(s)-f(0)$

Trigonometric identities:

$\sin(a\pm b)=\sin a\cos b\pm\cos a\sin b$

$\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$

$\cos^2 x=\frac{1}{2}(1+\cos 2x)$

$\sin^2 x=\frac{1}{2}(1-\cos 2x)$