Math 204: Differential Equations

Midterm Exam (A)

Sprint, 2019

NAME: Solutions

SECTION (check one box) : Section 1 (Inci 13:00) ☐
Section 2 (Kocyigit 13:00) ☐
Section 3 (Kocyigit 14:30) ☐

Instructions:

1. **Wait** for **signal** to begin.
2. **Write** your **name** in the space provided, and **check one box** to indicate which section of the course you belong to.
3. Please **turn off cell phones** or other electronic devices which may be disruptive.
4. Unless otherwise stated, you must **justify your solutions** to receive full credit. Work that is illegible may not be graded. Work that is scratched out will not be graded.
5. It is fine to leave your answer in a form such as $\ln(.02)$ or $\sqrt{239}$ or $(385)(13^3)$. However, if an expression can be easily simplified (such as $e^{\ln(.02)}$ or $\cos(\pi)$ or $(3-2)$), you should simplify it.
6. This exam is **closed book**. You may not use notes, or other external resource. You may use calculators.
   It is a violation of the honor code to give or receive help on this exam.
7. In True/False questions circle true, false or if you think the given information is not sufficient for a conclusion then write N/A and explain. Unless the question states "No explanation needed" you should explain your answer in order to get credit.
8. If you use a theorem you should check (and write clearly) whether the conditions of the theorem hold.

Problem	Score
1	5
2	9
3	5+8
4	10
5	10
6	5+10
7	15
8	5+6
9	12
Total	

1. Consider the differential equations for $y(x)$

$$L[y] = 0 \qquad (1)$$

$$L[y] = f(x) \qquad (2)$$

where L is a second order linear differential operator and assume all coefficients of L and f are differentiable functions.
Circle true or false for each statement. (No explanation needed.)

1pt each

[T / F] If y_p is a solution of (2) then Cy_p solves (2).

[T / F] If y_p is a solution of (2) and y_h is a solution to (1) then $y_p + Cy_h$ solves (2).

[T / F] If y_1 and y_2 are solutions of (2) then $y_1 - y_2$ solves (1).

[T / F] If y_1 and y_2 are solutions of (2) then $y_1 + Cy_2$ solves (1) for any constant C.

[T / F] There exists a constant solution of $y' = y(x - y)$

2. Given the below 3 equations

$$t^2y' + 2ty + e^t = 0 \qquad (1)$$

$$3y' + 4y + 5t = 0 \qquad (2)$$

$$\left(1+t^2\right)y' + \left(1-e^t\right)\left(1+y^2\right) = 0 \qquad (3)$$

Circle true or false. (No explanation needed, DE stands for differential equation)

1pt each

[T / F] DE (1) is exact [T / F] DE (1) is linear [T / F] DE (1) is separable

[T / F] DE (2) is exact [T / F] DE (2) is linear [T / F] DE (2) is separable

[T / F] DE (3) is exact [T / F] DE (3) is linear [T / F] DE (3) is separable

3. (a) Find the general solution of

$$y'' + 3y' + 2y = 0$$

Answer: $y(t) = c_1 e^{-t} + c_2 e^{-2t}$

char. eqn. $r^2 + 3r + 2 = 0$ $r_1 = -1$ $r_2 = 2$

e^{-t} and e^{-2t} are soln.

+5

(b) Solve the IVP

$$y'' + 3y' + 2y = 5e^{-2t}, \qquad y(0) = 2, \quad y'(0) = -8$$

Answer: $y(t) = e^{-t} + e^{-2t} - 5te^{-2t}$

Undetermined coeff. $Y_p = At^s e^{-2t}$

duplication e^{-2t} solves homogeneous eq. $\Rightarrow s = 1$.

Candidate: $Y_p = Ate^{-2t}$,

$Y_p' = Ae^{-2t} - 2Ate^{-2t}$, $Y_p'' = -4Ae^{-2t} + 4Ate^{-2t}$

Substituting: $-4Ae^{-2t} + \cancel{4Ate^{-2t}} + 3Ae^{-2t} - \cancel{6Ate^{-2t}} + \cancel{2Ate^{-2t}} = 5e^{-2t}$

$\Rightarrow A = -5$.

+6

gen soln $y(t) = c_1 e^{-t} + c_2 e^{-2t} - 5te^{-2t}$

IC: $2 = c_1 + c_2$

$-8 = -c_1 - 2c_2 - 5e^0$

$\Rightarrow -1 = -c_2 \Rightarrow c_1 = 1, \ c_2 = 1$

+2

4. Find general solution of

$$ty' + 2y = t^2.$$

Answer: $y(t) = t^2/4 + c/t^2$, $t \neq 0$ (or $t>0$ also ok)

$\boxed{\text{first order linear}}$ $\boxed{y' + \frac{2}{t} y = t}$ Integrating factor $\mu = e^{\int 2/t} = e^{2\ln|t|} = e^{\ln t^2} = t^2$

$t>0$ or $t<0$

+5

$\Rightarrow t^2 y' + 2ty = t^3 \Rightarrow (t^2 y)' = t^3 \Rightarrow t^2 y = \frac{t^4}{4} + c$

$\Rightarrow y(t) = \frac{t^2}{4} + \frac{c}{t^2}$

+5

5. Solve the following equation. (You can give an implicit solution).

$$y' = -\frac{x+2y}{2x+3y}$$

Answer: $\frac{x^2}{2} + 2xy + \frac{3}{2}y^2 = C$

+2

$\Rightarrow \underbrace{(x+2y)}_{M} + \underbrace{(2x+3y)}_{N}y' = 0$ $\quad M_y = 2 = N_x \Rightarrow$ DE is $\underline{\text{exact}}$

+4

$\psi_x = x + 2y$ and $\psi_y = 2x+3y$

+2

$\Rightarrow \psi(x,y) = \frac{x^2}{2} + 2xy + f(y)$ $\quad \psi(x,y) = 2xy + \frac{3y^2}{2} + g(x)$

$\Rightarrow f(y) = \frac{3y^2}{2}$ and $g(x) = \frac{x^2}{2}$ $\Rightarrow \psi = \frac{x^2}{2} + 2xy + \frac{3y^2}{2}$

+2

implicit soln is $\psi(x,y) = C$.

6. Find the general solution of each of the below differential equations.

(a) $y'' + 4y' + 4y = 0, \quad t > 0$

Answer: $y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

+4

+1

$r^2 + 4r + 4 = 0 \Rightarrow r_1 = r_2 = -2$

e^{-2t}, te^{-2t} are soln.

(b) $y'' + 4y' + 4y = t^{-1}e^{-2t}, \quad t > 0$

Answer: $y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + t(\ln t) e^{-2t}$

+2

Variation of parameters $y_p = u_1 e^{-2t} + u_2 t e^{-2t}$

+2

$$\begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t^{-1}e^{-2t} \end{pmatrix}$$

$$W = e^{-4t} - \cancel{2te^{-4t}} + \cancel{2te^{-4t}} = e^{-4t}.$$

+2

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = e^{4t} \begin{pmatrix} \alpha & -te^{-2t} \\ \alpha & e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ t^{-1}e^{-2t} \end{pmatrix} = e^{4t} \begin{pmatrix} -e^{-4t} \\ t^{-1}e^{-4t} \end{pmatrix} = \begin{pmatrix} -1 \\ t^{-1} \end{pmatrix}$$

$u_1' = -1 \Rightarrow u_1 = -t$ $\qquad u_2' = t^{-1} \Rightarrow u_2 = \ln t$

+4

$y_p = -te^{-2t} + (\ln t)te^{-2t}$

note that te^{-2t} solves homogeneous eqn thus we can simplify y_p [why? see first question]

$y_p = t(\ln t)e^{-2t}$

or: $y_p = -y_1 \cdot \int \frac{y_2 \cdot g}{\underbrace{W}_{1}} dt + y_2 \cdot \int \frac{y_1 \cdot g}{\underbrace{W}_{t^{-1}}} dt$

7. $y_1(t) = e^t$ is a solution of the differential equation

$$(t-1)\,y'' - ty' + y = 0, \quad t > 1.$$

Find a fundamental solution set. (Hint: There might be a solution in the form $u(t)\,e^t$)

Answer: $\{t, e^t\}$

$$y'' - \frac{t}{t-1}y' + \frac{1}{t-1}y = 0$$

For reduction of order set $y = u(t)e^t$.

Then $p = \dfrac{-t}{(t-1)}$

$$e^t u'' + \left(2e^t - \frac{t}{t-1}e^t\right)u' = 0$$

Note:

If you don't remember this, then following hint substituting ue^t in DE gives same eqn.

$$u'' + \left(1 - \frac{1}{t-1}\right)u' = 0$$

+7

setting $w = u'$: $\dfrac{w'}{w} = -1 + \dfrac{1}{t-1}$

$$\ln|w| = -t + \ln(t-1) \qquad t > 1$$

$$w = e^{-t}e^{\ln(t-1)} = (t-1)e^{-t}$$

$$u = \int w = \int te^{-t} - e^{-t} = -te^{-t}$$

$$\Rightarrow y = -te^{-t}\cdot e^{t} = -t$$

+8

m

m

8. Answer each.

(a) Find Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2\pi \\ 0, & 2\pi \leq t < \infty \end{cases}$$

Answer: $\left(1 - e^{-2\pi s}\right)/s$

$$\mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^{2\pi} e^{-st}\,dt + 0$$

$$= \frac{e^{-st}}{-s}\Big|_0^{2\pi} = \frac{-e^{-2\pi s}}{s} + \frac{1}{s}$$

$$= \frac{1-e^{-2\pi s}}{s}$$

+5

(b) Laplace transform of $\sin(at)$ is $\frac{a}{s^2+a^2}$, $s > 0$ and Laplace transform of $\cos(at)$ is $\frac{s}{s^2+a^2}$, $s > 0$. Find a function whose Laplace transform is

$$\frac{9-5s}{s^2+4}, \quad s > 0$$

Answer: $\frac{9}{2}\sin(2t) - 5\cos(2t)$

$$\frac{9-5s}{s^2+4} = \frac{9}{2} \cdot \frac{2}{s^2+4} - 5\,\frac{s}{s^2+4}$$

using linearity of $\mathcal{L}$

$$\mathcal{L}\left(\frac{9}{2}\sin(2t) - 5\cos(2t)\right) = \frac{9}{2}\mathcal{L}(\sin(2t)) - 5\,\mathcal{L}(\cos(2t))$$

$$= \frac{9}{2} \cdot \frac{2}{s^2+4} - 5\,\frac{s}{s^2+4}$$

+6

9. **Circle** true or false. If true then **show**, if false then **explain** or give a counter example, if the statement can't be concluded from the given information write **N/A** and explain.

[T / F] If y_1 and y_2 are solutions of the equation

$$y' = cos(t^2)y + t$$

and if $y_1(t_0) = y_2(t_0)$ at some point t_0 then $y_1(t) = y_2(t)$ for all t.
Explain:

Set $y_0 = y_1(t_0) = y_2(t_0)$
Then IVP with initial condition $y(t_0) = y_0$
has unique soln in $\mathbb{R}$ since $\cos t^2$ and t are
continuous on whole $\mathbb{R}$. But y_1 and y_2 solves
this IVP. Therefore $y_1 = y_2$. by uniqueness of IVP.

+4

[T / F] There are infinitely many solutions of the equation

$$y'' + ty' + sin(t)y = e^{t^2}$$

which passes through origin.
Explain:

Consider IVP with initial condition $y(0)=0$ $y'(0)=A$
This IVP has unique solution since $t, \sin t, t^2$ are continuous
[by second order uniq. & existence thm].
Thus for each $A \in \mathbb{R}$ there is a different solution
that passes through origin (i.e. $y(0)=0$)

+4

[T / F] There exists a unique function that satisfies the IVP

$$y' = \frac{cos(ty)}{t}, \qquad y(1) = 0$$

on some interval $1 - h < t < 1 + h$.
Explain:

$y' = f(t,y)$ where $f(t,y) = \frac{\cos(ty)}{t}$

Both f, f_y are continuous everywhere except $t=0$.

+4

Take $R = (1/2, 3/2) \times (-1,1)$, a rectangle where f and f_y
are continuous. and initial point $(1,0)$ is in R.
Thus uniqueness follows from first order non-linear theorem.

10 / 10