1 Stochastic Differential Equations Driven by fBm: Distributional and Path Properties

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Fractional Brownian motion (fBm) has been introduced by Kolmogorov to study the turbulence in an incompressible fluid flow . Long range dependence and self-similarity properties of fBm announced itself in the Nile river level studies of Hurst The parameter $H \in (0,1)$ which characterizes fBm W_t^H $\left(t \geq 0\right)$ is called Hurst parameter. Mandelbrot and van Ness were first to study the properties of the fBm. They have also shown that fBm might find an application in finance. Standard Brownian motion becomes a special case of the fBm with the Hurst parameter $H = 1/2$. This property alone arises great interest in the literature. In contrast with the standard Brownian motion, increments of W_t^H $\left(t \geq 0\right)$ are no longer statistically independent for nonoverlapping intervals of t. We now witness the foot prints of the fBm W_t^H $\left(t \geq 0\right)$for $H \in (1/2, 1)$ ranging from the characters of the solar activity to weather derivatives in mathematical finance. An fBm W_t^H $\left(t \geq 0\right)$ is neither a semi-martingale nor a Markov process. Therefore, a new stochastic calculus is needed for its treatment. Recently, C. Bender and B. Oksendal have developed a new calculus which is valid for $H \in (0,1)$.

In this talk, we consider Itô stochastic ordinary differential equation (SODE) of the form

$$dx_i = f_i(x,t)dt + g_{i\alpha}(x,t)dW_\alpha^H, \ \ 1 \leq i \leq n; 1 \leq \alpha \leq r, \tag{1}$$

where $f_i(x,t)$ is a drift vector and $g_{i\alpha}(x,t)$ is a diffusion matrix and, dW_α^H is the infinitesimal increment of fBm (summation convention applies to repeated indices and drops for indices in paranthesis hereafter). Based on an Itô lemma by Bender and Oksendal we derive the corresponding Fokker Planck equation. Furthermore Liouville theorem has been extended to Itô SODE given in equation (1). Finally we have obtained analytical solutions to the FPK equation via conserved quantities of the deterministic. This result leads to the fluctuaution-dissipation theorem. Furthermore, we seek for linearizing transformations for the one-dimensional, nonlinear Itô SODE given in (1). We find two linearization criteria for (1) to be linearizable thereby extending the Gard theorem to SDEs driven by fBm. Furthermore, stochastic integrating factor has been introduced and used to solve the linear Itô SDE for $H \in (0, 1)$. We show that nonlinear mean-reverting Ornstein-Uhlenbeck equation, Cox-Ingersol-Ross model and population growth model in noisy environment are linearizable when they are driven by fBm. Analytical (explicit) solutions to these models are given. We have developed a modified Milstein numerical scheme for Itô SDE given in (**??**) in Section 5. Finally we comaper the numerical solutions with the analytical solutions obtained via linearization.