Inversion Formulas in Truncated Data Ray-Tomography

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Abstract

In Ray-Tomography the goal is to recover a function in higher dimensions from knowledge of integrals of the function along lines. For example, in the Euclidean setting we let $f \in C_0^\infty(\mathbb{R}^2)$, and for each straight line L in $\mathbb{R}^2$, we define the Ray-Transform of f along L as

$$Rf(L) := \int_L f(l)dl \qquad (dl = \text{Lebesgue measure in } L).$$

The inversion problem consists of recovering f at all or some points of $\mathbb{R}^2$ from knowledge of Rf along all or some of the straight lines in $\mathbb{R}^2$. Or if $\mathbb{H}$ is the Hyperbolic plane and $f : \mathbb{H} \to \mathbb{R}$ is smooth and compactly supported, for each $\gamma : \mathbb{R} \to \mathbb{H}$ arc-length parametrized geodesic we define the Ray-Transform of f along γ as

$$Rf(\gamma) := \int_{\mathbb{R}} f(\gamma(s))ds.$$

Again, the inversion problem consists of recovering f at all or some points of $\mathbb{H}$ from knowledge of Rf along all or some geodesics in $\mathbb{H}$.

These two particular cases of Ray-Tomography are closely related to applications in medical imaging techniques, like Computed Tomography or Positron Emission Tomography [3], as well as to geological exploration techniques, appearing as the linearized problem in Travel Time Tomography [7].

The ray transform has been amply studied in general settings [8]. In addition to injectivity, support theorems and stability results for particular cases, inversion formulas have also been provided [2, 3, 5, 6]. Unfortunately, these inversion formulas require knowledge of Rf over all straight lines (or geodesics) to recover f at one given point, which is very inconvenient from the point of view of applications.

A different approach for the Euclidean case was proposed in [4]. The inversion of the Ray-Transform is reduced to a problem of inverting a one dimensional Hilbert transform. A new inversion formula, not requiring knowledge of all of Rf, was obtained (eg. [4]) and injectivity and stability results for other cases of Euclidean truncated measurements followed [1].

In this talk we will first present how the Ray-Transform appears as a model of the measurements in the applications mentioned above. Then we will overview the non-locality issue of the previous inversion formulas. Afterwards we will go into more detail about the approach for the Euclidean setting introduced in [4], with the generalizations obtained in [1]. To conclude, we will present some new results generalizing the approach in [4] to the Hyperbolic plane setting.

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