KOÇ UNIVERSITY

MATH 106

FIRST MIDTERM November 11, 2013

Duration of Exam: 75 minutes

INSTRUCTIONS:

- No calculators may be used on the test.
- No books, no notes, no questions, and no talking allowed.
- You must always explain your answers and SHOW YOUR WORK to receive full credit.
- Print (use CAPITAL LETTERS) and sign your name. GOOD LUCK!

SURNAME, Name: KEY

Student ID no: ____________________

Signature: ____________________

(Check One):

Lecture 1 (Haluk Oral – MW 14:00-15:15) : ___
Lecture 2 (Selda Küçükçifçi – MW 11:00-12:15) : ___
Lecture 3 (Selda Küçükçifçi – MW 12:30-13:45) : ___
Lecture 4 (Haluk Oral – MW 15:30-16:45) : ___

PROBLEM	1	2	3	4	5	TOTAL
POINTS	30	15	16	24	15	100
SCORE						

Problem 1. Find the following limits, if they exist. Do not use l'Hospital's rule.

(a) **(6 pts)** $\lim\limits_{x\to -1}\left(\dfrac{x-2}{x^2-x-2}\right)$ $= \lim\limits_{x\to -1} \dfrac{(x-2)}{(x-2)(x+1)} = \lim\limits_{x\to -1} \dfrac{1}{x+1}$

$\lim\limits_{x\to -1^-} \dfrac{1}{x+1} = -\infty$

$\lim\limits_{x\to -1^+} \dfrac{1}{x+1} = +\infty$

(b) **(6 pts)** $\lim\limits_{x\to\infty}[\ln(x^2-1)-\ln(x+1)]$ $= \lim\limits_{x\to\infty} \ln \dfrac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}}$

$= \ln\left(\lim\limits_{x\to\infty}(x-1)\right) = \infty$

(c) **(6 pts)** $\lim\limits_{x\to 0}\left[\ln\left(\dfrac{x\sin x}{1-\cos x}\right)\right]$ $= \ln\left(\lim\limits_{x\to 0} \dfrac{(x\sin x)(1+\cos x)}{1-\cos^2 x}\right)$

$= \ln\left(\lim\limits_{x\to 0} \dfrac{x}{\sin x}\cdot(1+\cos x)\right) = \ln\left[\underbrace{\lim\limits_{x\to 0}\dfrac{x}{\sin x}}_{1}\cdot \underbrace{\lim\limits_{x\to 0}(1+\cos x)}_{2}\right] = \ln 2.$

(d) **(6 pts)** $\lim\limits_{x\to\infty}\dfrac{\sin 2x}{x}$

$-1 \le \sin 2x \le 1$

$-\dfrac{1}{x} \le \dfrac{\sin 2x}{x} \le \dfrac{1}{x}$

$\lim\limits_{x\to\infty} -\dfrac{1}{x} = \lim\limits_{x\to\infty}\dfrac{1}{x} = 0$ So $\lim\limits_{x\to\infty}\dfrac{\sin 2x}{x} = 0.$

(e) **(6 pts)** $\lim_{x\to 0}\left(\frac{e^{3x}-1}{x}\right)$ $= f'(0)$ where $f(x) = e^{3x}$

$\Rightarrow f'(x) = 3e^{3x}$

$f'(0) = 3$

So $\lim_{x\to 0} \frac{e^{3x}-1}{x} = 3$

Problem 2. (15 pts) Using the ϵ, δ definition of the limit prove that

$$\lim_{x\to 3}(4x+1) = 13.$$

For every $\varepsilon > 0$ we need to find a $\delta > 0$ such that

$|x-3| < \delta \Rightarrow |4x+1-13| < \varepsilon.$

Choose $\delta = \frac{\varepsilon}{4}$ then

if $|x-3| < \delta = \frac{\varepsilon}{4}$ then $4|x-3| < \varepsilon$

which is $|4x-12| = |4x+1-13| < \varepsilon.$

Problem 3 (a) (12 pts) Prove that the equation $\tan^{-1} x = \frac{\pi}{2} - x$ has at least one real root. ($\tan^{-1} x = \arctan x$)

Let $f(x) = \tan^{-1} x + x - \frac{\pi}{2}$

$f(x)$ is continuous everywhere.

Consider $f(0) = \tan^{-1} 0 + 0 - \frac{\pi}{2} = -\frac{\pi}{2} < 0$

& $f(1) = \tan^{-1} 1 + 1 - \frac{\pi}{2} = \frac{\pi}{4} - \frac{\pi}{2} + 1 = 1 - \frac{\pi}{4} > 0$

Since $f(0) < 0 < f(1)$

there is $c \in (0,1)$ such that $f(c) = 0$

by the Intermediate Value Theorem.

Hence $\tan^{-1} c + c - \frac{\pi}{2} = 0$ for some $c \in (0,1)$.

(b) (4 pts) State the theorem you use in part (a).

Let f be a continuous function on $[a,b]$ and let N be a number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number $c \in (a,b)$ such that $f(c) = N$.

Problem 4 (24 pts) Find the derivative of the function f in (a) – (b).

(a) **(6 pts)** $f(x) = (x^3 - 2x)^{\ln x}$ $= y$

$$\ln y = \ln x \ln(x^3 - 2x)$$

$$\frac{y'}{y} = \frac{1}{x}\ln(x^3-2x) + (\ln x)\frac{3x^2-2}{x^3-2x}$$

$$y' = \left(\frac{\ln(x^3-2x)}{x} + \frac{(3x^2-2)\ln x}{x^3-2x}\right)(x^3-2x)^{\ln x}$$

(b) **(6 pts)** $f(x) = \arcsin(\sin x)$, where $x \in (\pi, 3\pi/2)$.

$$f'(x) = \frac{1}{\sqrt{1-\sin^2 x}} \cdot \cos x = \frac{\cos x}{|\cos x|} = \frac{\cos x}{-\cos x} = -1.$$

(c) **(6 pts)** Determine y', where $\frac{y^3}{x^2} = 1 + 3^{3y}$ by using implicit differentiation.

$$\frac{3y^2y'x^2 - y^3 2x}{x^4} = 3^{3y} \cdot 3 \cdot \ln 3 \cdot y'$$

$$y'(3y^2x^2 - (3\ln 3)3^{3y}x^4) = 2y^3x$$

$$y' = \frac{2y^3x}{3y^2x^2 - (3\ln 3)3^{3y}x^4}$$

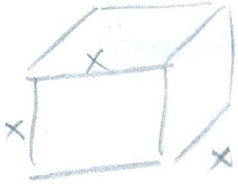
(d) **(6 pts)** Find a solution for $f(x)$ if $\dfrac{d}{dx}(\sin(f(x)) = \dfrac{\cos(f(x))}{x}$.

$$\frac{d}{dx}\left(\sin(f(x))\right) = \cos(f(x)) \cdot f'(x) = \frac{\cos(f(x))}{x}$$

So $f'(x) = \dfrac{1}{x}$

Then $f(x) = \ln x$ satisfies the above equality.

Problem 5 (15 pts) The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?



$V = x^3$

$A = 6x^2$

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt} = 10\ cm^3/min \Rightarrow \frac{dx}{dt} = \frac{10}{3x^2}$$

$$\left.\frac{dA}{dt}\right|_{x=30} = 12x \left.\frac{dx}{dt}\right|_{x=30} = 12 \cdot \cancel{30} \cdot \frac{10}{3 \cdot \cancel{30}^{2}} = \frac{4}{3}\ cm/min.$$