KOÇ UNIVERSITY

MATH 106 - CALCULUS 1

Final Exam June 5, 2015

Duration of Exam: 105 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign your name, and indicate your section below.**

Name: ______________________

Surname: KEY

Signature: ______________________

Section (Check One):

Section 1: Selda Küçükçifçi M-W (8:30) —
Section 2: Ayberk Zeytin T-Th(10:00) —

PROBLEM	POINTS	SCORE
1	40	
2	27	
3	20	
4	18	
TOTAL	105	

1. Let $f(x) = \dfrac{1}{(1+e^x)^2}$.

(a) (2 points) Find the domain of f.

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(b) (4 points) Find x and y-intercepts (if they exist) of f.

$x=0$ $y=\frac{1}{4}$ $(0, \frac{1}{4})$ y-intercept.

no x-intercept.

(c) (6 points) Find horizontal and vertical asymptotes (if they exist) of f.

$\lim\limits_{x\to\infty} \dfrac{1}{(1+e^x)^2} = 0$ $y=0$ is a horizontal asymptote

$\lim\limits_{x\to-\infty} \dfrac{1}{(1+e^x)^2} = 1$ $y=1$ " " " "

no vertical asymptote.

(d) (10 points) Find intervals on which f is increasing or decreasing and determine critical points if they exist.

$f(x) = (1+e^x)^{-2}$ $f'(x) = -2(1+e^x)^{-3} e^x$

$= \dfrac{-2e^x}{(1+e^x)^3} < 0$

So f is decreasing everywhere.

There is no critical point.

(e) (10 points) Find intervals on which f is concave up or down and determine inflection points if they exist. (Hint: Consider $\ln 2 \sim 0.7$ and $4/9 \sim 0.44$.)

$$f''(x) = \frac{-2e^x(1+e^x)^3 + 2e^x \cdot 3(1+e^x)^2 \cdot e^x}{(1+e^x)^6} = \frac{2e^x(1+e^x)^2(3e^x-1-e^x)}{(1+e^x)^6}$$

$$= \frac{2e^x(2e^x-1)}{(1+e^x)^4}$$

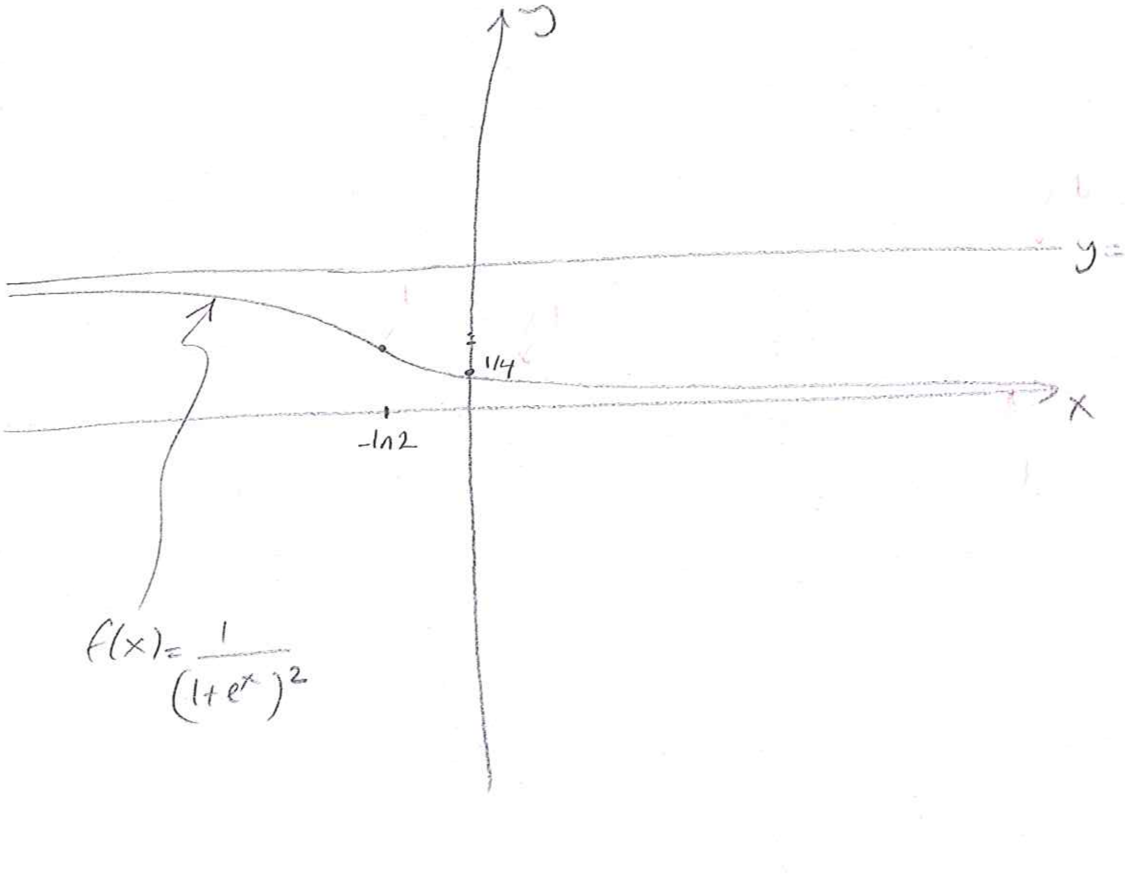
$$f''(0) = 0 \Rightarrow e^x = 1/2 \Rightarrow x = \ln\left(\tfrac{1}{2}\right) = -\ln 2 \approx -0.7$$

	$-\ln 2$	
f''	$-$	$+$
f	concave down	concave up

inflection point : $\left(-\ln 2, f(-\ln 2)\right)$, $-\ln 2 \approx -0.7$, $f(-\ln 2) \approx 0.44$

$$f(-\ln 2) = \frac{1}{(1+e^{-\ln 2})^2} = \frac{1}{(1+\frac{1}{2})^2} = \frac{1}{\frac{9}{4}} = \frac{4}{9} \approx 0.44$$

(f) (8 points) Sketch the graph of f.



2. Test the following series for (conditional) convergence.

(a) (9 points) $\displaystyle\sum_{n=2}^{\infty} \frac{106n + \cos n + \sin n}{13n^2 - n}$

$\dfrac{106n + \cos n + \sin n}{13n^2 - n} > 0$ when $n \geq 2$.

$$\lim_{n\to\infty} \frac{\frac{106n + \cos n + \sin n}{13n^2 - n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{106n + \cos n + \sin n}{13n - 1}$$

$$-2 \leq \cos n + \sin n \leq 2$$

$$106n - 2 \leq 106n + \cos n + \sin n \leq 106n + 2$$

$$\frac{106n - 2}{13n - 1} \leq \frac{106n + \cos n + \sin n}{13n - 1} \leq \frac{106n + 2}{13n + 1}$$

$\frac{106n-2}{13n-1} \xrightarrow{n\to\infty} \frac{106}{13}$ and $\frac{106n+2}{13n+1} \xrightarrow{n\to\infty} \frac{106}{13}$

So $\displaystyle\lim_{n\to\infty} \frac{106n + \cos n + \sin n}{13n - 1} = \frac{106}{13}$

Since $\sum \frac{1}{n}$ is divergent then our series is divergent.

(b) (9 points) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{(-1)^n}{n}\right) = \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

This is an alternating series. So

1) $a_n \cdot a_{n+1} < 0$

also

2) $\displaystyle\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$ and 3) $a_{n+1} = \sin\left(\frac{1}{n+1}\right) < \sin\left(\frac{1}{n}\right) = a_n$

and

Since $n+1 > n$, $\frac{1}{n+1} < \frac{1}{n}$

So the series is convergent. But

$\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is divergent since $\displaystyle\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1$ and $\sum \frac{1}{n}$ is divergent.

Hence our series is conditionaly convergent.

(c) (9 points) $\sum_{n=1}^{\infty} \frac{106^n n!}{n^n}$

$$\frac{106^n n!}{n^n} > 0$$

$$\lim_{n\to\infty} \frac{\frac{106^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{106^n n!}{n^n}} = \lim_{n\to\infty} \frac{106(n+1)\, n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{106\, n^n}{(n+1)^n}$$

let $y = \left(\frac{n}{n+1}\right)^n$

$$\ln y = n \ln\left(\frac{n}{n+1}\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{(n+1-n)}{(n+1)^2}}{-\frac{1}{n^2}}$$

$$\frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{-n^2}{n(n+1)} = -1.$$

So $\lim_{n\to\infty} y = e^{-1}$ so $\lim_{n\to\infty} \frac{106 n^n}{(n+1)^n} = \frac{106}{e} > 1$.

Hence the series is divergent.

3. (10 points) Find the Taylor series of the function e^{2x+5} around $x = 3$.

$f(x) = e^{2x+5}$ $\qquad$ $f(3) = e^{11}$

$f'(x)\ 2e^{2x+5}$ $\qquad$ $f'(3) = 2e^{11}$

$f''(x) = 2^2 e^{2x+5}$ $\qquad$ $\vdots$

$f'''(x) = 2^3 e^{2x+5}$

$\vdots$

$f^{(k)}(x) = 2^k e^{2x+5}$ $\qquad$ $f^{(k)}(3) = 2^k e^{11}$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(3)}{k!}(x-3)^k = \sum_{k=0}^{\infty} \frac{2^k e^{11}}{k!}(x-3)^k.$$

(b) (5 points) Find the radius of convergence of the series you found in part (a).

$$L = \lim_{n\to\infty} \frac{2^{n+1} e^{11}}{(n+1)!} \cdot \frac{n!}{2^n e^{11}} = \lim_{n\to\infty} \frac{2}{n+1} = 0 \quad \text{so } R = \infty$$

(c) (5 points) Calculate the sum $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$.

let $\quad x = 2 \quad$ in $\quad e^{2x+5} = \sum_{k=0}^{\infty} \frac{2^k e^{11}}{k!}(x-3)^k$

$$e^9 = e^{11} \sum_{k=0}^{\infty} \frac{2^k (-1)^k}{k!}$$

So $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2} = \frac{1}{e^2}$.

4. (a) (9 points) Determine whether the following improper integral $\int_0^\infty \frac{dx}{xe^x}$ is convergent or divergent.

$$\int_0^\infty \frac{dx}{xe^x} = \int_0^1 \frac{dx}{xe^x} + \int_1^\infty \frac{dx}{xe^x}$$

$$\int_0^1 \frac{dx}{xe^x} \geq \frac{1}{e}\int_0^1 \frac{dx}{x} \quad \text{but} \quad \int_0^1 \frac{dx}{x} \text{ is divergent. So}$$

$$\int_0^1 \frac{dx}{xe^x} \text{ is divergent. Hence } \int_0^\infty \frac{dx}{xe^x} \text{ is divergent.}$$

(b) (9 points) Evaluate the integral $\int_1^e \sin(\ln x)dx$.

$$u = \sin(\ln x) \qquad dx = dv$$

$$du = \cos(\ln x)\cdot \frac{1}{x}\,dx \qquad x = v$$

$$I = \int_1^e \sin(\ln x)\,dx = \Big[x\sin(\ln x)\Big]_1^e - \underbrace{\int_1^e \cos(\ln x)\,dx}$$

$$u = \cos(\ln x) \qquad dx = dv$$

$$du = -\sin(\ln x)\cdot\frac{1}{x}\,dx \qquad x = v$$

$$I = \Big[x\sin(\ln x)\Big]_1^e - \Big[x\cos(\ln x)\Big]_1^e - \int_1^e \sin(\ln x)\,dx$$

$$2I = e\sin 1 - e\cos 1 + 1$$

$$I = \frac{1}{2}(e\sin 1 - e\cos 1 + 1).$$